Verifying Linearizability

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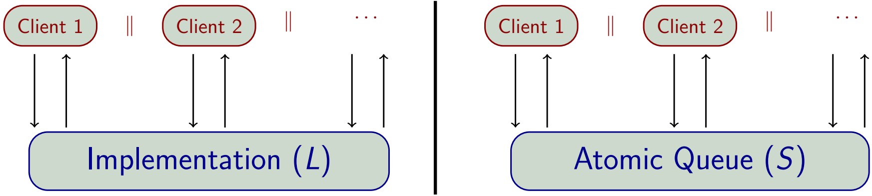
EPFL

19 December 2017

Why Linearizability? Ensuring Observational Refinement



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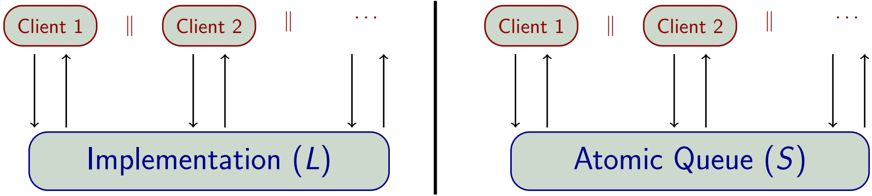

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$$\mathbf{P[L]} \subseteq \mathbf{P[S]}$$

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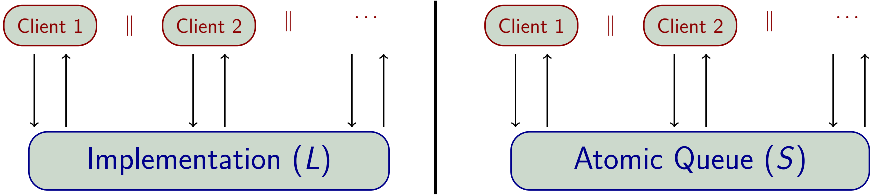

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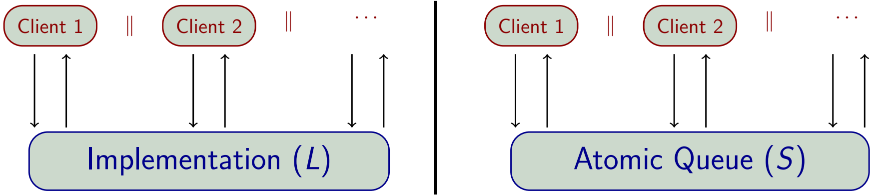

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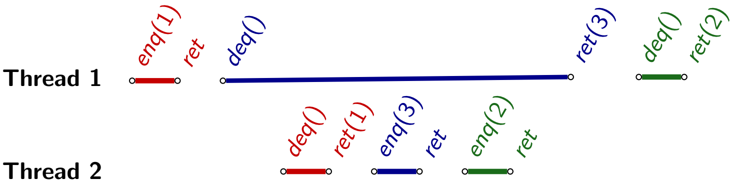
Application: If we prove a safety property on a program P using an atomic queue S, we can replace the atomic queue by a (more efficient) concurrent linearizable implementation L, and the safety property will still hold.

Events and Trace Example



Events and Traces

Definition (Events)

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Definition (Trace)

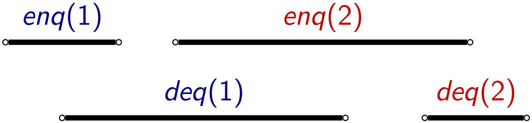
A **trace** is a sequence of call and return events.

Operation

Definition (Operation)

An **operation** is a tuple with a **thread identifier**, a **method name**, a **parameter** and a **return value**.
(corresponds to a pair of matching call and return events)

Linearization Points Example



$enq(1)$

$enq(2)$

$deq(1)$

$deq(2)$

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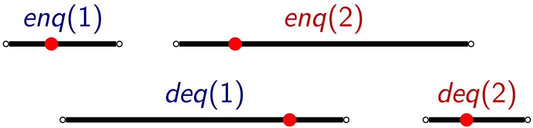

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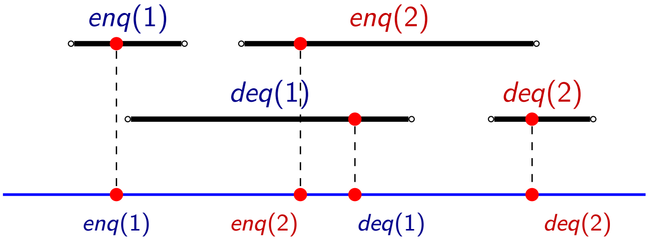

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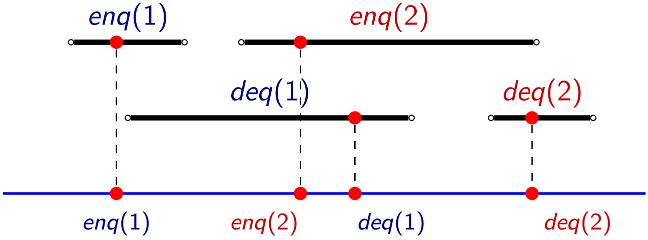

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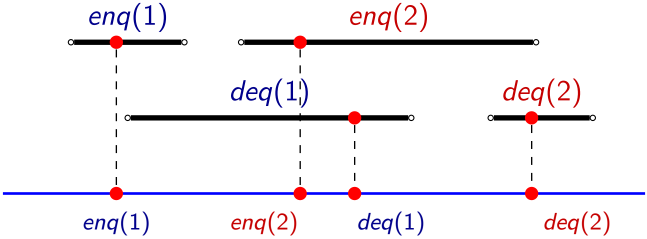

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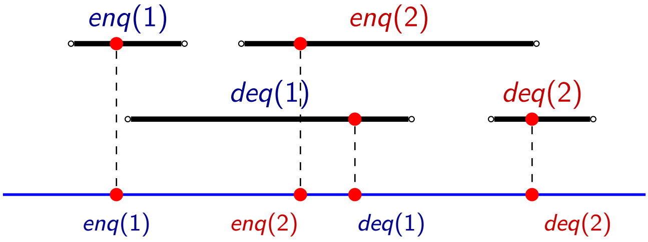

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And also linearizable to $deq(1) \cdot enq(1) \cdot enq(2) \cdot deq(2)$. (not a valid Queue sequence)

Linearization Points

Definition (Linearizability)

A trace t is **linearizable** with respect to a sequence of operations w, denoted $t \sqsubseteq w$ if, for each operation o, we can find a **point** (called linearization point) between the call and return event of o such that:
the obtained sequence of operations is w.

History

Definition (History)

A **history** $h = (O, <)$ is a strict partial order (irreflexive and transitive) over a set of operations O.

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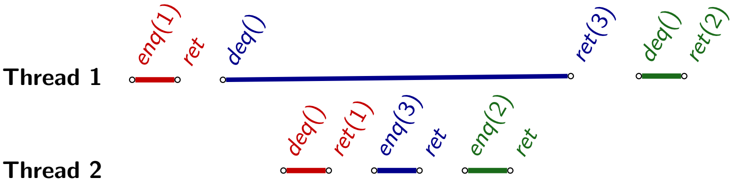
For a **trace** t, we define the history **hist(t)** to be $(O, <)$ where:

- O is the set of operations that appear in t
- for $o_1, o_2 \in O$, $o_1 < o_2$ iff the return event of o_1 is before the call event of o_2 in t.

Example of Trace/History



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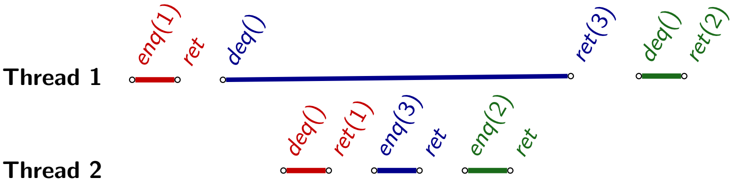

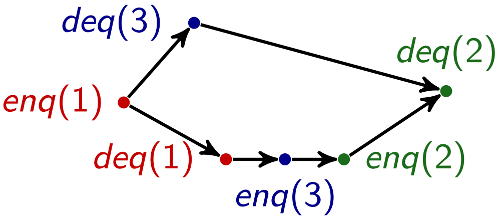

Another Definition for Linearizability

Definition (Linearizability of a history)

We say that a **history** $h = (O, <)$ is **linearizable** with respect to a sequence w, denoted $\mathbf{h} \sqsubseteq \mathbf{w}$ if we can obtain w by reordering the operations of h, while respecting the order $<$.

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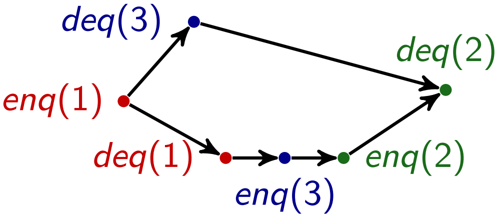
A **trace** t is **linearizable** with respect to w, denoted $t \sqsubseteq w$ if $hist(t)$ is **linearizable** with respect to w.

Example



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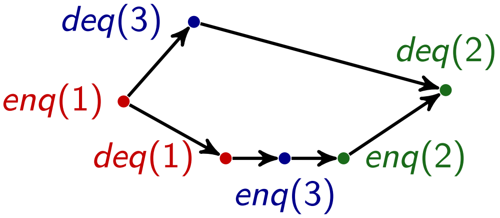

Linearizable to: $enq(1) \cdot deq(1) \cdot enq(3) \cdot deq(3) \cdot enq(2) \cdot deq(2)$.

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Definition (Linearizability of a library)

A **library** L is **linearizable** with respect to S, denoted $\mathbf{L} \sqsubseteq \mathbf{S}$ if every history/trace produced by L is linearizable with respect to S.

Linearizability checking problems

Problem (Testing)

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Linearizability checking problems

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Given a **history** h*, and a* **specification** S*, check whether* $h \sqsubseteq S$

Problem (Verification)

Given a **library** L*, and a* **specification** S*, check whether* $L \sqsubseteq S$*.*
(check $h \sqsubseteq S$ *for every* h *in* L*)*

Motivation for Testing: Bug-Finding

- Enumerate **many traces** of a library
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- Enumerate **many traces** of a library
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If we find a non-linearizable trace, we found a **bug**.

Limitation of testing: cannot verify that **all** the traces of a library are linearizable (there are infinitely many traces)

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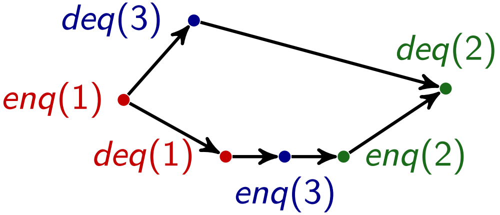
Worst case: $|O|!$ permutations to explore

Example



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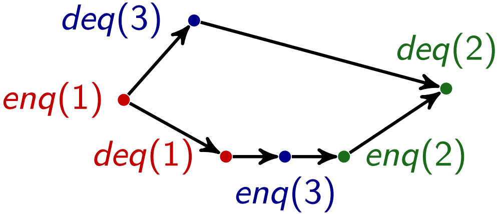
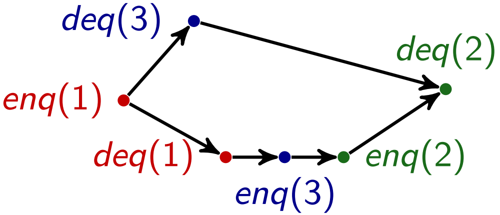


Check each of the $6! = 720$ permutations.

Example (minor improvement)



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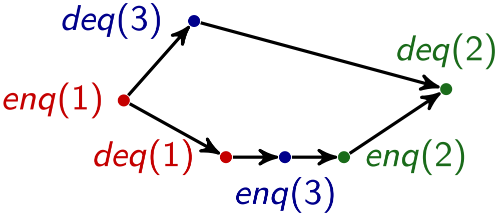

Start from the **minimal nodes**, and only explore linearizations that respect $<$ and the **specification**.

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For a history h, we have $h \sqsubseteq S$ iff **isLinearizable(Seq(), h)** holds.

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Polynomial-Time Algorithm for Testing?

Theorem (Gibbons & Korach 97)

Given h and S, checking $h \sqsubseteq S$ is NP-**complete**.

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(⇒) No **polynomial-time** algorithm, unless $P = NP$

Polynomial-Time Algorithm for Testing?

Theorem (Gibbons & Korach 97)

Given h and S, checking $h \sqsubseteq S$ is NP-**complete**.

($\Rightarrow$) No **polynomial-time** algorithm, unless $P = NP$

However, there are **polynomial-time** algorithms if we look at particular specifications S.

Testing Problem for Queues

Problem (Linearizability for Queues)

Given a **history** *h, check whether $h \sqsubseteq Queue$.*

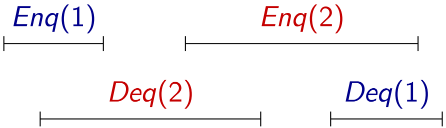
Bad Pattern 1 and Bad Pattern 1'

A dequeue operation with no corresponding enqueue.

- (BP1) a $deq(1)$ such that $enq(1)$ does not exist at all
- (BP1') two or more $deq(1)$ (this is bad because we assume enqueues are unique)

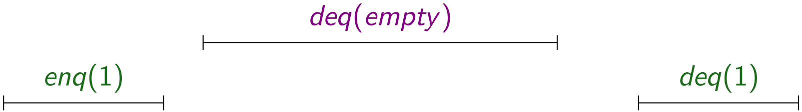
Bad Pattern 2

Two enqueue's $enq(1) < enq(2)$ such that $deq(2) < deq(1)$.
(if $deq(1)$ isn't in the history, we pose that $deq(2) < deq(1)$ holds)



Bad Pattern 3 (Example A)

A *deq*(*empty*) operation **covered** by pairs of enqueue/dequeue.



Bad Pattern 3 (Example B)

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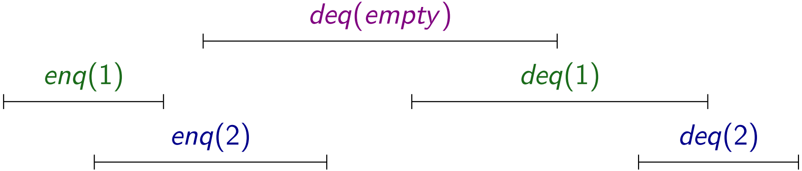

deq(*empty*)

enq(1)

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enq(2)

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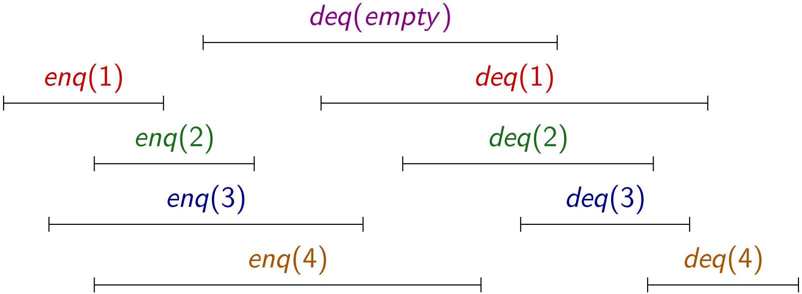
Bad Pattern 3 (Example C)

A *deq*(*empty*) operation **covered** by pairs of enqueue/dequeue.



Defining Bad Pattern 3 Formally

Given a history $h = (O, <)$, and some $deq(empty)$ operation in O, we construct a graph G such that:

- the vertices of G are the **values** that are enqueued in h and a vertice for the $deq(empty)$ operation
- there is an edge from v_1 to v_2 iff $enq(v_1) < deq(v_2)$
- there is an edge from $deq(empty)$ to v iff $deq(empty) < deq(v)$
- there is an edge from v to $deq(empty)$ iff $enq(v) < deq(empty)$

Definition

The operation $deq(empty)$ is **covered** iff there is a **cycle** going through $deq(empty)$ in the graph.

Bad Patterns

- (BP1) a *deq*(v) such that there exists no *enq*(v)
- (BP1') two *deq*(v) operations (or more)
- (BP2) two enqueue operations dequeued in the **wrong order**
- (BP3) a *deq*(*empty*) operation which is **covered**

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Theorem (Bad Patterns)

Let h be a history (with unique enqueues).
Then $h \sqsubseteq Queue$ **if and only if**
h doesn't contain one of these **bad patterns**

Polynomial-time algorithm

We can check in polynomial-time if h has a bad pattern.

Theorem

Let h be a history (with unique enqueues).
We can check $h \sqsubseteq Queue$ in **polynomial-time**.

Proof.

Check for the absence of bad patterns. Each one can be checked in polynomial-time.

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□

Limitations of Testing

Checking that $h \sqsubseteq S$ one by one, we can never be sure that $L \sqsubseteq S$
A library produces an **infinite** amount of traces/histories.

Herlihy & Wing Queue

```
var table = Map[Int,Value]()     // represents the queue
var n: Int = 0                   // index of the next enqueue
```

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def enqueue(v: Value): Unit = {
  synchronized { i = n; n = n + 1 } // atomic operation
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Herlihy & Wing Queue

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def dequeue(): Value = {
  while(true) {
    val m = n
    for (k <- 0 to m-1) {
      // get the element at index k, and write null instead
      val v = SWAP(table(k), null)
      // if not null, return the element
      if (v != null)
        return v
    }
  }
}
```

H&W Queue is Linearizable

Theorem

The H&W Queue $L_{H\&W}$ *is linearizable, i.e.* $L_{H\&W} \sqsubseteq Queue$.

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Proof.

We prove that $h \sqsubseteq Queue$ for every $h \in L_{H\&W}$. It suffices to prove that h has no bad pattern. We assume that h has a bad pattern and derive a contradiction (stetch)

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- BP1': Not possible when assuming unique enqueues, and due to the atomicity of SWAP.

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- BP2: Not possible as the *first* enqueue operation will be stored at a smaller index in the table
- BP3: Not possible because dequeue never returns empty

□

Summary

- Testing for **finding bugs**
- Verification for finding bugs or proving correctness
- Checking linearizability for one trace is NP-**complete**
- But, **polynomial-time** if we restrict the specification to Queue/Stack and histories with unique enqueues/pushes
- It is enough to check for **bad patterns**
- Careful: Stack bad patterns are **not symmetric** wrt Queue bad patterns

References:

(1) Aspect-Oriented Linearizability Proofs. Chakraborty et al.
(2) On Reducing Linearizability to State Reachability. Bouajjani et al.