Self-stabilizing node coloring

Context

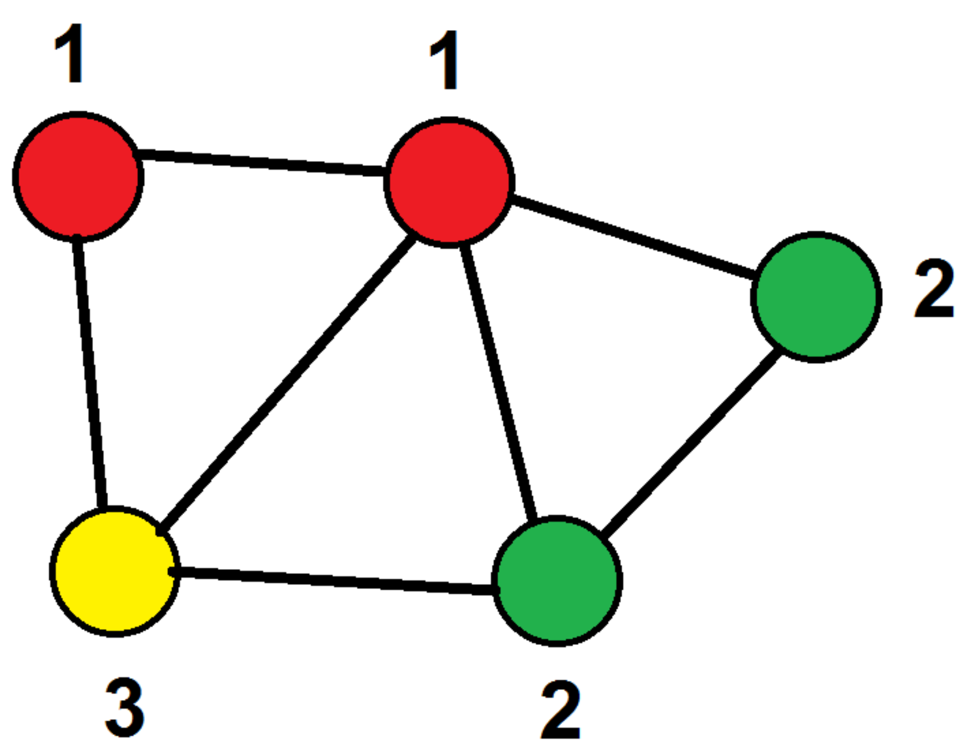
- A graph of degree D
(D = max number of neighbors per node)

- D + 1 "colors" {1,2,...,D}

- Each node p has a color $C(p) \in \{1,2,...,D\}$

Node coloring problem

Initially, the nodes have any colors:

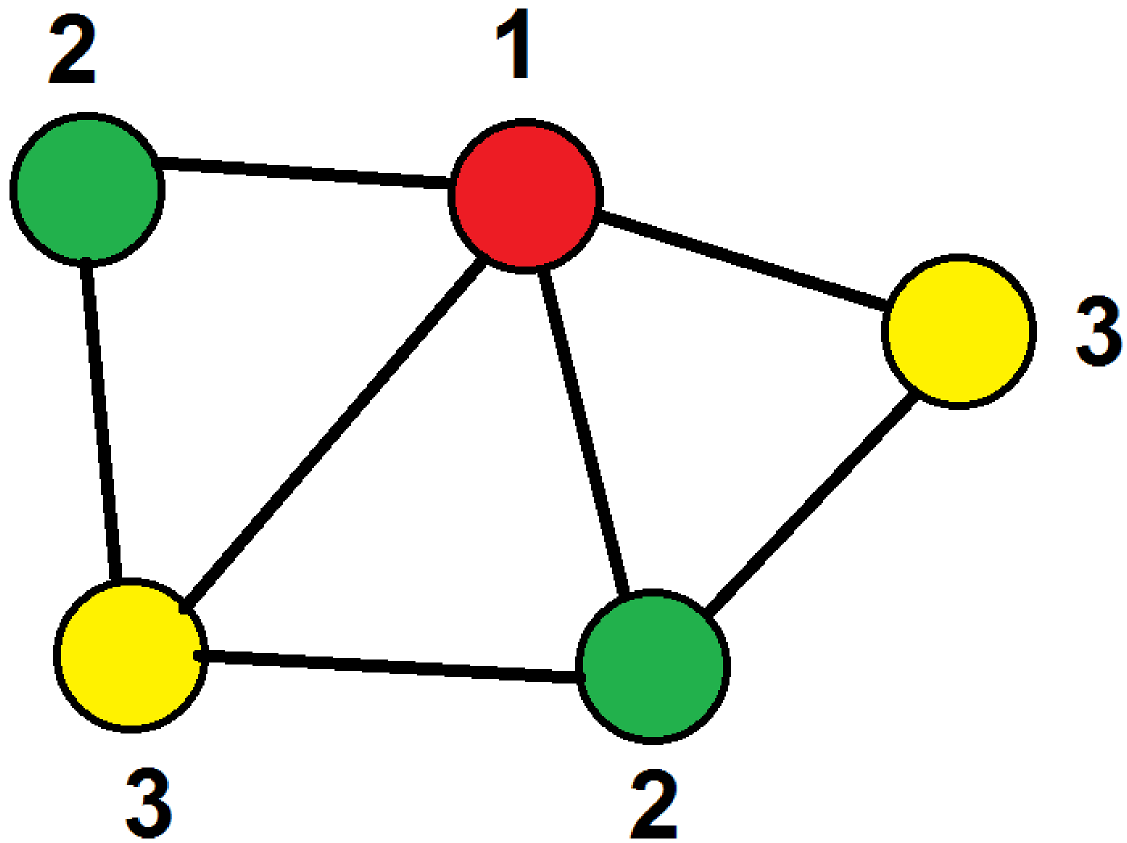


Eventually, we must satisfy the following property:

For any two neighbor nodes p and q, C(p) ≠ C(q)

(the graph is **"well colored"**)

Example of "well colored" graph:



Model

- Each node is eventually "activated"
- When a node is "activated", it can execute a given algorithm
- Two neighbor nodes are never activated at the same time

Algorithm

When a node p is activated :

- Let N(p) be the set of neighbors of p
- Let C be a color such that :

$$\forall q \in N(p), C(q) \neq C$$

→ Then, C(p) := C

(Such a color C always exists because:
- p has at most D neighbors
- we have D+1 colors)

Our goal

Prove that, with this algorithm,
the graph is always eventually
"well colored",
AND remains "well colored"

(In other words, the coloring of the graph is <u>**self-stabilizing**</u>, because it works for any initial coloring)

Definition

A node p is "well colored" if:

$$\forall q \in N(p), C(q) \neq C(p)$$

$\rightarrow$ If all nodes are "well colored", then the graph is "well colored"

Lemma 1 (Liveness property)

Let p be a node that is
"not well colored"*.*

Then, p is eventually ***"well colored"****.*

Proof

- p is eventually activated

- When p is activated, no neighbor of p is activated in the same time

- Then, p executes the algorithm, and takes a color different from its neighbors

- Then, p becomes **"well colored"**

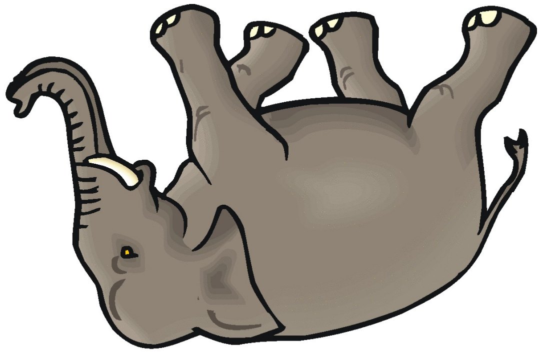
Lemma 2 (Safety property)

If p is ***"well colored"****, then p always remains* ***"well colored"****.*

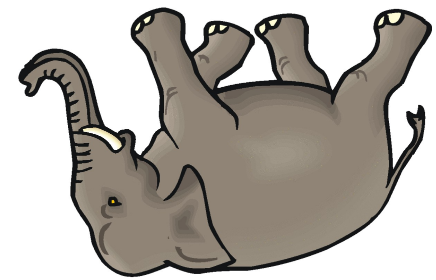
Proof

The proof is by contradiction:

- We suppose the opposite



- We show that this leads to a contradiction



Suppose the opposite: a node p is **"well colored"**, then, after a certain time, p is **"not well colored"**.

→ Changes only happen when nodes are activated.
Therefore, consider the activation where p goes from **"well colored"** to **"not well colored"**.

→ 2 cases (mutually exclusive) :

- p is activated
- at least one neighbor of p is activated

Case 1: p is activated

By hypothesis, no neighbor of p is activated at the same time.

Then, it implies that p takes the same color as one of its neighbors.
→ **contradiction with the algorithm!**

Case 2: at least one neighbor q of p is activated

By hypothesis, p is not activated at the same time.

Then, it implies that q takes the same color as p.
→ **contradiction with the algorithm!**

Liveness property:
*A node **"not well colored"** eventually becomes **"well colored"**.*

Safety property:
*A node **"well colored"** always remains **"well colored"**.*

→ **Each node** is eventually **"well colored"**, and remains **"well colored"**

→ **The graph** is eventually **"well colored"**, and remains **"well colored"**

We proved that this (simple) algorithm is **self-stabilizing** for the node coloring problem.