Population protocols

Consider population comprising two species of agents σ_g (green) and σ_b (blue). We study the following predicate:

$$P: \quad \#\sigma_g - 2 \cdot \#\sigma_b \leq 4 \tag{1}$$

where $\#\sigma_c$ denotes the number of agents with color c. Recall the protocol $\mathcal{A}$ defined in the lecture.

- state space: leader bit $l \in \{L, \perp\}$; counter $u \in \{-s, \dots, s\}$ where $s \geq 5$ is a fixed constant; output bit $b \in \{0, 1\}$.
- initialization: leader bit $l_{init} = 1$ (all leaders); counter $u_{init} = 1$ if green agent, $u_{init} = -2$ otherwise (blue agent); output bit $b_{init} = 0$
- rules: when two agents x, y meet, $(l_x, u_x, b_x),\ (l_y, u_y, b_y) \to (l'_x, u'_x, b'_x),\ (l'_y, u'_y, b'_y)$
  - if both non-leaders ($l_x = l_y = \perp$), then nothing changes.
  - if, e.g., agent x is a leader ($l_x = L$), then

$$\begin{aligned} l'_x &= L, \quad l'_y = \perp \\ u'_x &= q(u_x, u_y) \underset{def}{=} \max\{-s, \min\{s, u_x + u_y\}\} \\ u'_y &= r(u_x, u_y) \underset{def}{=} u_x + u_y - q(u_x, u_y) \\ b'_x &= b'_y = b_x \end{aligned}$$

We assume the following fairness condition.

Definition 1 (Global fairness). *An execution E is fair if and only if for every configuration C occurring infinitely often in E, for every configuration C' reachable from C, C' also occurs infinitely often in E.*

Intuitively, it means that if something is reachable infinitely often, then it is actually reached infinitely often.

The goal of this exercice is to prove that $\mathcal{A}$ computes the predicate P. First, some warm-up.

Question 1. What does it mean for a population protocol to compute the predicate P?

Question 2. Show that in any fair execution of $\mathcal{A}$, there is eventually a single leader.

Question 3. Show that, for any configuration C in an execution,

$$\sum_{\text{agent } x} u_x(C) = \#\sigma_g - 2 \cdot \#\sigma_b$$

where $u_x(C)$ is the value of the counter of agent x in configuration C.

Thanks to the previous claims, we can focus on the suffix E' (of the execution E) in which there is a single leader λ. We have to prove that, eventually, the counter u_λ of λ satisfies:

$$u_\lambda = \max\{-s, \min\{s, \#\sigma_g - 2 \cdot \#\sigma_b\}\} \tag{2}$$

The proof relies on the (classical) *potential method*. For any configuration C in the suffix E', consider the quantity

$$p(C) = \sum_{x \neq \lambda} |u_x(C)| \tag{3}$$

Intuitively, this function measures the (non-negative) “mass” of the non-leaders. We will show that p cannot increase, and thus, is eventually constant.

Consider a transition $C \to C'$ in the execution suffix E' due to the meeting of the leader λ and (non-leader) agent x. We use the following notations:

$$\begin{aligned} u_\lambda = u_\lambda(C) \ u'_\lambda = u_\lambda(C') \\ u_x = u_x(C) \ u'_x = u_x(C') \end{aligned}$$

Question 4. Assume that $u_x \geq 0$. Show that

$$u'_\lambda = u_\lambda + \min\{u_x, s - u_\lambda\}$$
$$u'_x = u_x - \min\{u_x, s - u_\lambda\}$$

Conclude that, in this case, p does not increase during the transition.

Question 5. Show that p does not increase either if $u_x \leq 0$.

We conclude that p is eventually constant (non-increasing sequence of integer values). Let E'' be the suffix (of E') during which p is constant.

Question 6. For any configuration C in E'' show that, if it is impossible decrease p from C, then one of the following cases holds:

- $p(C) = 0$
- $u_\lambda(C) = s$ and for any $x \neq \lambda$, $u_x(C) \geq 0$
- $u_\lambda(C) = -s$ and for any $x \neq \lambda$, $u_x(C) \leq 0$

Conclude that

$$u_\lambda(C) = \max\{-s, \min\{s, \#\sigma_g - 2 \cdot \#\sigma_b\}\}$$

Question 7. Show that $\mathcal{A}$ computes the predicate P. *(don't forget the fairness assumption)*