Distributed Algorithms

Fall 2019

Links & Gossip
2nd exercise session, 30/09/2019

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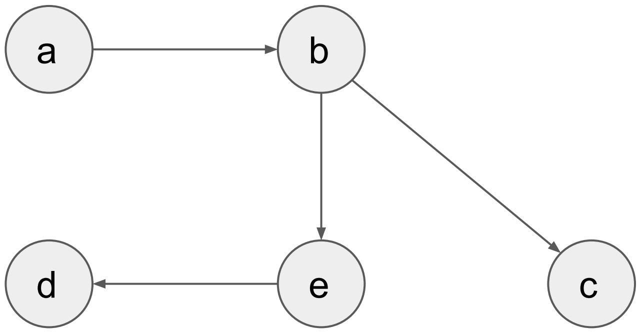
Graphs

A **graph** is a couple *(V, E)* where *V* is a set of **vertices** and $E \subseteq V^2$ is a set of **edges**.



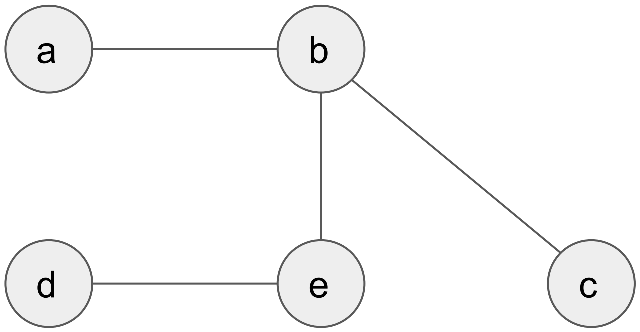
Example graph (V, E):

- V = {a, b, c, d, e}
- E = {(a, b), (b, c), (b, e), (e, d)}

Two vertices are **adjacent** (or **neighbors**) iff an edge exists between them. In the example, *a* and *b* are adjacent; *a* and *d* are not adjacent.

Graphs (undirected)

An **undirected graph** is a graph *(V, E)* such that *(a, b)* ∈ *E* if and only if *(b, a)* ∈ *E*.



Example graph (V, E):

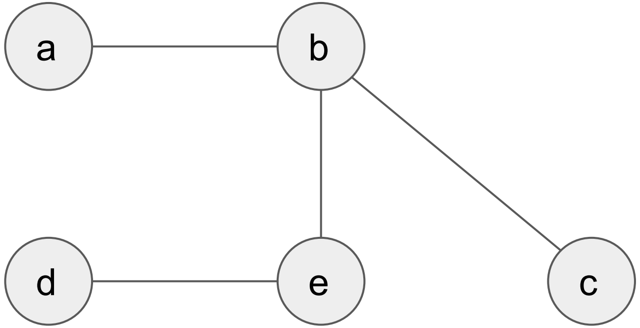
- V = {a, b, c, d, e}
- E = {(a, b), (b, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e)}

We use undirected graphs to model networks of processes:

- Each vertex represents a process
- Two vertices are neighbors iff the corresponding processes can directly exchange messages.

Paths

A **path** is a sequence of *distinct* vertices $(v_1, \ldots, v_N)$ such that, for all $i \in [1, N - 1]$, v_i and v_{i+1} are adjacent.



Some paths in (V, E):

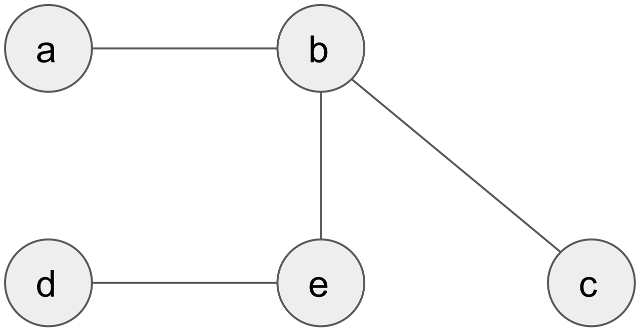
- (a, b)
- (a, b, c)
- (a, b, e, d)

While

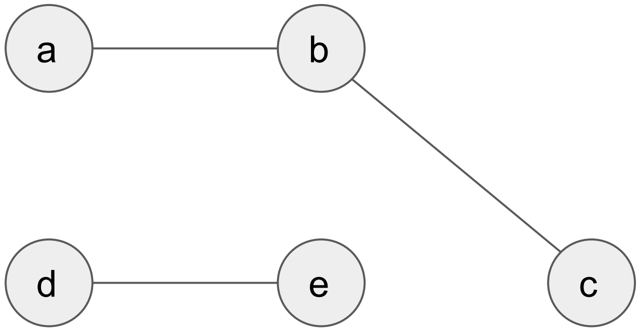
- (a, c, e) *is **not** a path:* a *and* c *are not adjacent!*

Connectivity

Two distinct vertices *a* and *z* are **connected** if and only if at least one path *(a, …, z)* exists in the graph. A graph is connected if any two distinct vertices are connected.



A connected graph



A disconnected graph

Exercise 1 (connectivity)

Prove that **connectivity** is a **symmetric property** on an undirected graph: let *a, b* be vertices such that *a* is connected with *b*. Prove that *b* is connected with *a*.

Hint: you can do it constructively.

Exercise 1 (solution)

- If *a* is connected to *b*, then a path *p* exists from *a* to *b*.
  Let $p = (a, v_1, \ldots, v_N, b)$.

- Since the graph is undirected, if *v* is adjacent to *w*, then *w* is adjacent to *v*.

- Therefore, the sequence $p' = (b, v_N, \ldots, v_1, a)$ is also a path.

- Since *p'* begins in *b* and ends in *a*, a path exists between *b* and *a*.
  Consequently, *b* is connected to *a*.

Exercise 2 (connectivity)

Prove that **connectivity** is a **transitive property** on an undirected graph: let *a, b, c* be vertices such that *a* is connected with *b* and *b* is connected with *c*. Prove that *a* is connected with *c*.

Hint: double-check the definition of a path.

Exercise 2 (solution)

- Let $p = (v_1, \ldots, v_N)$ and $q = (w_1, \ldots, w_M)$ be the paths from a to b and from b to c, respectively. We have $v_1 = a$, $v_N = w_1 = b$, $w_M = c$.
- We note that $(v_1, \ldots, v_N, w_2, \ldots, w_M)$ is in general not a path, as the vertices are not guaranteed to be disjoint.
- If $a \in q$, then a and c are trivially connected. Indeed, a subpath $q' = (w_K, \ldots, w_M)$ already exists in q such that $w_K = a$ and $w_M = c$.
- If $\neg(a \in q)$, then let $v_K = w_H$ be the first element of p that is also in q. Since $v_N = w_1 = b$, v_K is guaranteed to exist.
- By definition, $v_1, \ldots, v_{K-1}$ are not in q. Therefore, $r = (v_1, \ldots, v_K, w_{H+1}, \ldots, w_M)$ is a path.
- Since r begins in a and ends in c, a and c are connected.

Exercise 3 (connectivity)

Write a procedure (pseudocode or any programming language) that inputs an undirected graph G = *(V, E)* and outputs *true* if and only if the *G* is connected.

Hint: use the results from Exercises 1 and 2.

Exercise 3 (solution)

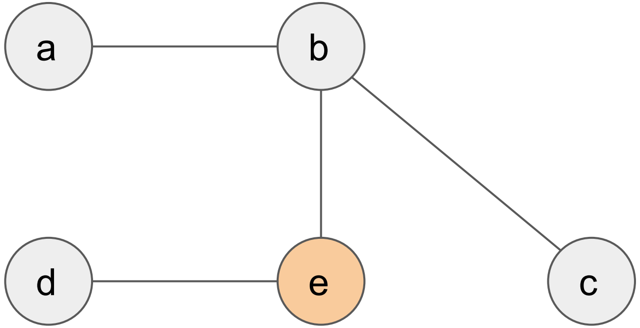
We start by noting that, since connectivity is symmetric and transitive, we only need to check if *any* node is connected to every other. We can implement the following algorithm:

- Pick any vertex v from V. Initialize a frontier set $F = \{v\}$. Initialize an interior set $I = \varnothing$.
- Until F is empty:
  - Pick an element f from F. Remove f from F, add f to I.
  - For every neighbor n of f:
    - If $\neg(n \in F \cup I)$, add n to F.
- If $I = V$, then G is connected.

Let w be any vertex, if and only if path exists between v and w, then w is eventually added to F, then removed from F and added to I. If eventually $I = V$, then every vertex is connected to v, and consequently G is connected.

Gossip

We use an undirected graph to represent which processes can communicate.
Upon receiving a new message *m*, a process forwards *m* to all its neighbors.



Example: diffusion of a message m *from process* e.

- e *issues* m

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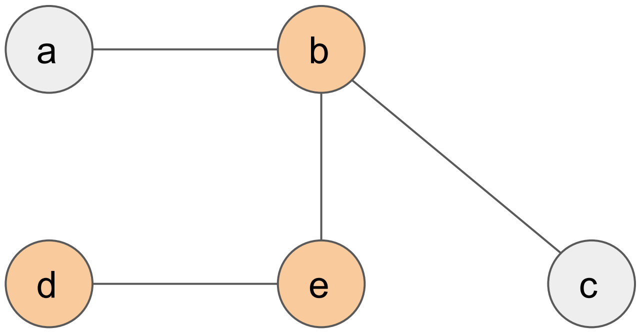

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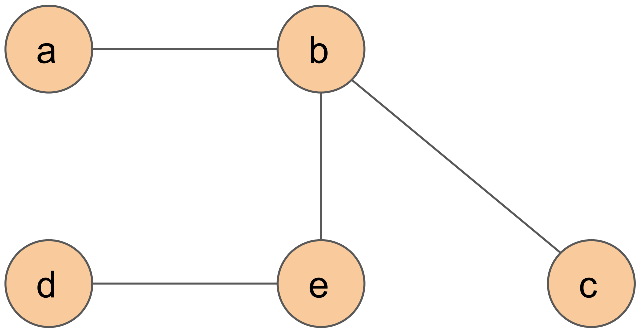


Example: diffusion of a message m *from process* e.

- e *issues* m.
- b *and* d *receive* m.
- *a and* c *receive* m.

Gossip is **correct** if and only if, if the sender is correct, every correct process eventually receives the message.

Exercise 4 (gossip)

Prove that gossip is correct if and only if the subgraph of correct processes is connected.

Note: prove **both** directions of the implication!

Hint: induction is your friend.

Exercise 4 (solution)

If the subgraph of correct processes is connected, then gossip is correct.

Let $G = (V, E)$ be the gossip network, let $N = |V|$, let s be the sender. By induction:

- Let s be the sender. We obviously have that s eventually delivers the message m.
- Let V_L denote the set of vertices that are connected to s by a path no longer than L. We have $V_0 = \{s\}$.
- Let N_L denote the set of vertices that have at least one neighbor in V_L. If every process in V_L eventually delivers m, then also every process in N_L delivers m (as m is sent to every neighbor).
- Since $N_L \cup V_L = V_{L+1}$, if every process in V_L eventually delivers m then every process in V_{L+1} eventually delivers m.
- Since all the vertices in a path are distinct, no path longer than N can exist on the gossip path. Therefore, $V = V_N$. Consequently, every node in V eventually delivers m.

Exercise 4 (solution)

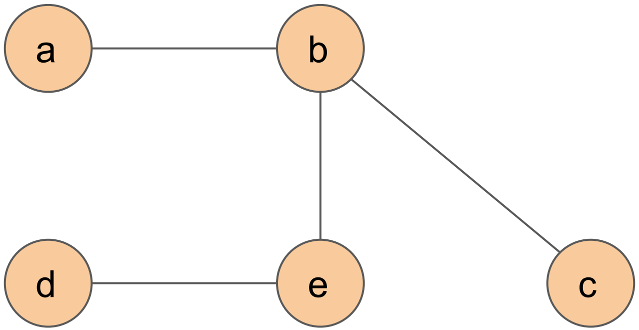
If gossip is correct, then the subgraph of correct processes is connected.

Let $G = (V, E)$ be the gossip network, let $N = |V|$, let s be the sender.

- Let $v \neq s$ be a correct process. Regardless of the crashes, v eventually delivers m. Therefore, v eventually receives m from a correct process.
- We use induction similarly to the previous slide, defining W_L as the set of processes that are connected to v by a path not longer than L.
- Let $i \in [0, N]$. If W_i includes s, then v is connected to s.
- If W_i does not include s, then at least one process in W_i eventually receives m from one of its neighbors, and that neighbor is not in W_i.
- Since the size of W_i is strictly increasing until W_i includes s, we have that W_N must include s.
- Since this holds true for every v, every process is connected to s, making the subgraph of correct processes connected.

Exercise 5 (gossip)

In the following system, exactly one process crashes. What is the minimum number of edges we need to add so that gossip is always correct?



Exercise 5 (solution)

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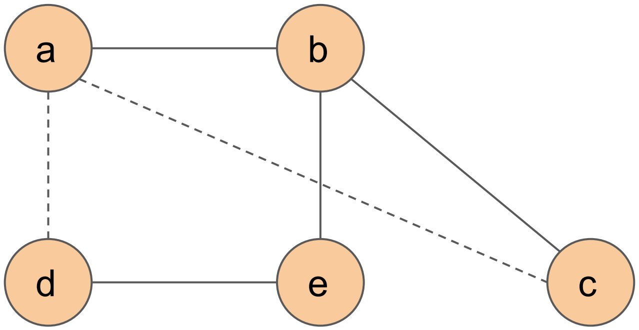


k-connectivity

Two paths *p*, *p'* connecting two vertices *a* and *z* are **disjoint** if they have no vertex in common, except *a* and *z*:

$$p = (a, b, \ldots, y, z)$$

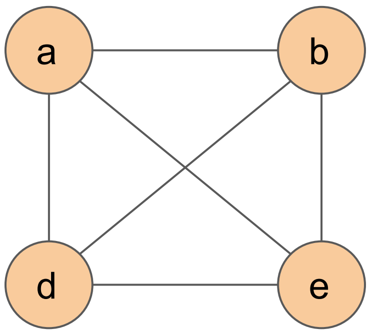
$$p' = (a, b', \ldots, y', z)$$

$$\{a, b, \ldots, y, z\} \cap \{a, b', \ldots, y', z\} = \{a, z\}$$

A graph is ***k*-connected** if and only if *k* disjoint paths exist between any two vertices of the graph.

Robustness

Gossip is **robust** to *k* failures if and only if it is always correct, as long as no more than *k* nodes are crashed.



A fully connected gossip graph is robust to N failures, where N is the number of processes.

Exercise 6 (robustness)

Prove that, if the gossip graph is *(k+1)*-connected, then gossip is *k*-robust.

Is the converse also true? Find a counterexample if not.

Hint: contradiction is your friend.

Exercise 6 (solution)

- By contradiction, let us assume that gossip is *$(k + 1)$*-connected, but k processes exist such that, if they all crash, then two correct processes a and b are no longer connected.
- By hypothesis, $(k + 1)$ distinct paths $p_1, \ldots, p_{k+1}$ exist between a and b.
- If some i exists such that no process crashes in p_i, then a and b are still connected by correct processes, and (as we proved in Exercise 4) they can gossip with each other.
- Since $p_1, \ldots, p_{k+1}$ are all distinct, at least one distinct process must crash in each p_i for a and b to be disconnected. But at most k processes can crash!

Exercise 6 (solution)

Technically:

But does it still work for $N > 2$?

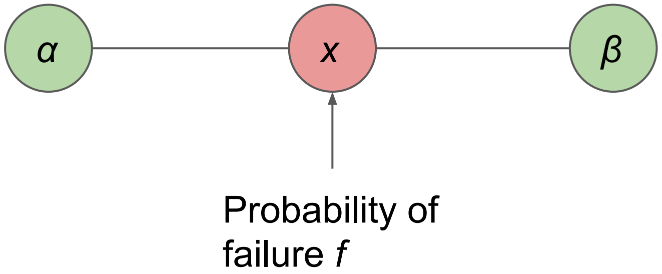
Random failures

Suppose that processes can fail independently with probability *f.*

What is the probability that two *correct* processes can communicate in the presence of failures?

It depends on their connectivity!

e.g.



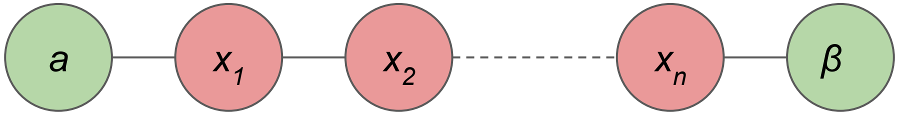
α, β can communicate iff x has not failed =>

α, β communicate with probability $1-f$.

Exercise 7 (random failures on series topology)

Suppose that processes x_i, $i=1, \dots, n$ can fail independently with probability f.

What is the probability that a and b can communicate?



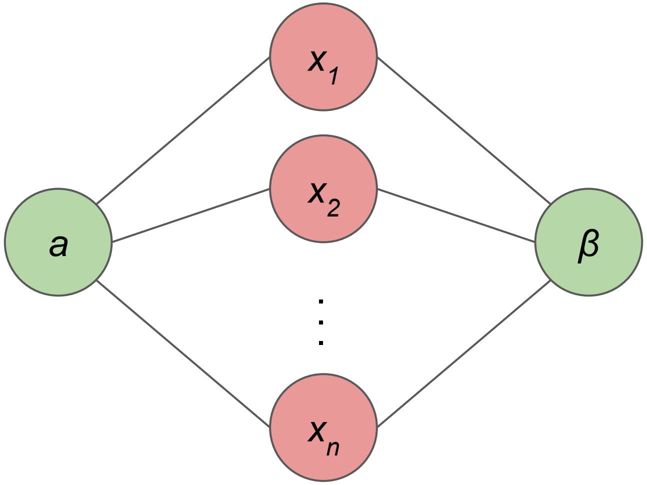
Exercise 7 (solution)

- Each process survives (i.e., it does not fail) with independent probability $(1 - f)$.
- Therefore, all processes survive with probability $(1 - f)^n$.

Exercise 8 (random failures on parallel topology)

Suppose that processes x_i, $i=1, \ldots, n$ can fail independently with probability f.

What is the probability that a and b can communicate?



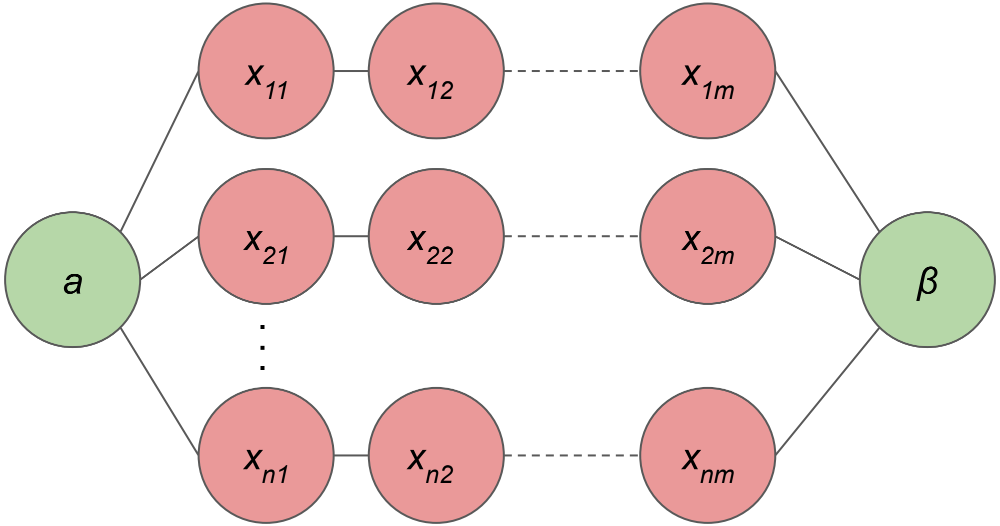
Exercise 8 (solution)

- Each process fails with independent probability f.
- Therefore, all processes fail with probability f^n.
- Finally, at least one process survives with probability $(1 - f^n)$.

Exercise 9 (random failures on series/parallel topology)

Suppose that processes x_{ij}, $i=1, \ldots, n$, $j=1, \ldots, m$ can fail independently with probability f.

Prove that a and b can communicate with probability $1 - [1 - (1-f)^m]^n$.

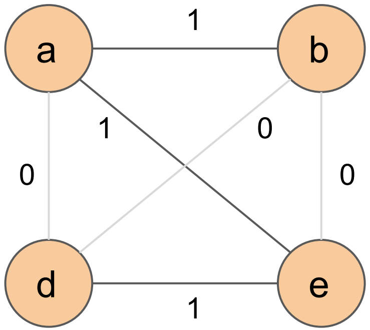


Exercise 9 (solution)

- As we proved in Exercise 7, every *branch* fails with independent probability $g = 1 - (1 - f)^m$.
- We can now consider each *branch* as if it was one of the processes in Exercise 8. The probability that no branch fails is $1 - g^n = 1 - [1 - (1-f)^m]^n$.

Erdös-Renyi graphs

An Erdös-Renyi graph $G(N, p)$ is a random undirected graph with N vertices, such that any two distinct vertices have an independent probability p of being adjacent.



Example graph
G(4, ½)

An Erdös-Renyi graph is defined by the values of $N(N-1)/2$ independent Bernoulli random variables:

$$E_{ij} \sim Bernoulli(p)$$
$$E_{ij} = E_{ji}$$

with $i, j \in V$. Vertices i and j are adjacent iff $E_{ij} = 1$.

Bonus Exercise 10 (Erdös-Renyi graphs)

What distribution underlies the number of edges in an Erdös-Renyi $G(N, p)$?
What distribution underlies the degree (i.e., number of links) of any vertex?
Are the degrees of any two vertices independently distributed?

Hint: how is the sum of Bernoulli variables distributed?

Connectivity of *G(N, p)*

Let *C(N, p)* denote the probability of a random graph *G(N, p)* being connected.
It is possible to prove that:

$$\lim_{N\to\infty} G(N, p) = 0 \qquad \text{iff } p < \ln(N) / N$$
$$\lim_{N\to\infty} G(N, p) = 1 \qquad \text{iff } p > \ln(N) / N$$

A large Erdös-Renyi graph is almost surely connected, as long as each vertex has an expected degree larger than *ln(N)*.

We can use Erdös-Renyi graphs to build probabilistic gossip with *logarithmic communication complexity*!

Bonus Exercise 11 (Erdös-Renyi graphs)

Write a distributed procedure that runs on N processes to build an Erdös-Renyi graph $G(N, ln(N)/N)$. We assume no failures. Each process can invoke:

- A procedure *rand(x)* that returns a real number between 0 and x, independently picked with uniform probability.
- A procedure *connect(i)* to connect to the i-th process.

Is it possible for the procedure to have $O(ln(N))$ computation complexity?