Distributed Algorithms

Fall 2019

Links & Gossip
2nd exercise session, 30/09/2019

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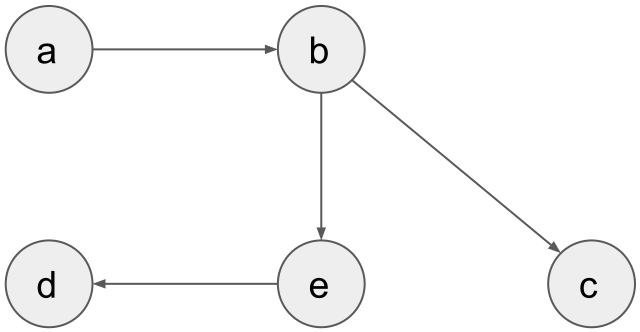
Graphs

A **graph** is a couple *(V, E)* where *V* is a set of **vertices** and $E \subseteq V^2$ is a set of **edges**.



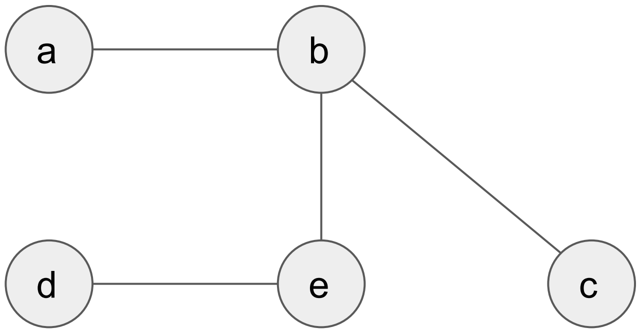
Example graph (V, E):

- V = {a, b, c, d, e}
- E = {(a, b), (b, c), (b, e), (e, d)}

Two vertices are **adjacent** (or **neighbors**) iff an edge exists between them. In the example, *a* and *b* are adjacent; *a* and *d* are not adjacent.

Graphs (undirected)

An **undirected graph** is a graph *(V, E)* such that *(a, b)* $\in$ *E* if and only if *(b, a)* $\in$ *E*.



Example graph (V, E):

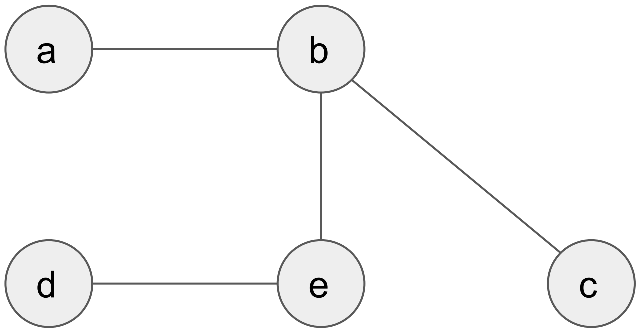
- V = {a, b, c, d, e}
- E = {(a, b), (b, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e)}

We use undirected graphs to model networks of processes:

- Each vertex represents a process
- Two vertices are neighbors iff the corresponding processes can directly exchange messages.

Paths

A **path** is a sequence of *distinct* vertices $(v_1, \ldots, v_N)$ such that, for all $i \in [1, N - 1]$, v_i and v_{i+1} are adjacent.



Some paths in (V, E):

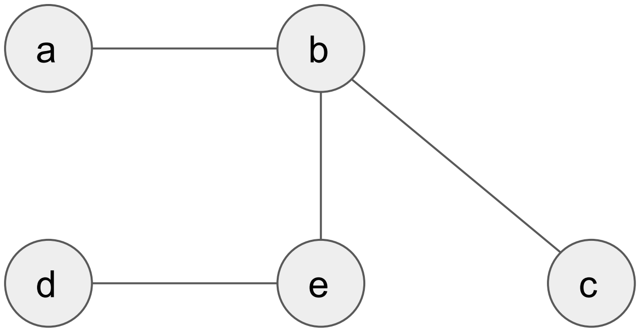
- (a, b)
- (a, b, c)
- (a, b, e, d)

While

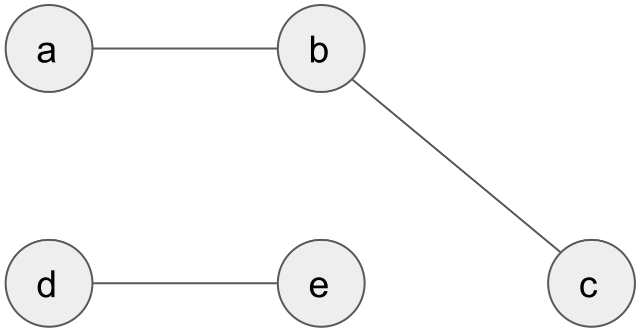
- (a, c, e) *is* ***not*** *a path:* a *and* c *are not adjacent!*

Connectivity

Two distinct vertices *a* and *z* are **connected** if and only if at least one path *(a, …, z)* exists in the graph. A graph is connected if any two distinct vertices are connected.



A connected graph



A disconnected graph

Exercise 1 (connectivity)

Prove that **connectivity** is a **symmetric property** on an undirected graph: let *a, b* be vertices such that *a* is connected with *b*. Prove that *b* is connected with *a*.

Hint: you can do it constructively.

Exercise 2 (connectivity)

Prove that **connectivity** is a **transitive property** on an undirected graph: let *a, b, c* be vertices such that *a* is connected with *b* and *b* is connected with *c*. Prove that *a* is connected with *c*.

Hint: double-check the definition of a path.

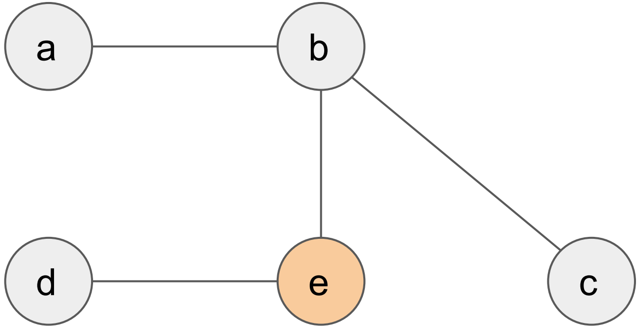
Exercise 3 (connectivity)

Write a procedure (pseudocode or any programming language) that inputs an undirected graph G = *(V, E)* and outputs *true* if and only if the *G* is connected.

Hint: use the results from Exercises 1 and 2.

Gossip

We use an undirected graph to represent which processes can communicate.
Upon receiving a new message *m*, a process forwards *m* to all its neighbors.



Example: diffusion of a message m *from process* e.

- e *issues* m

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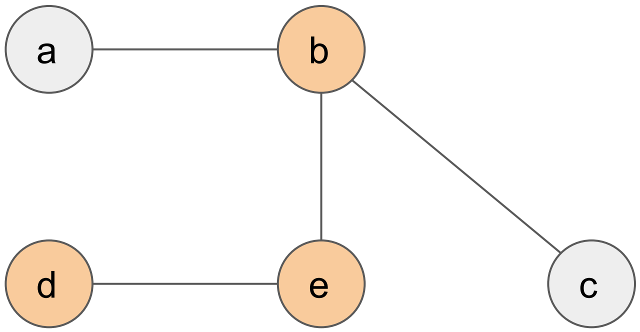

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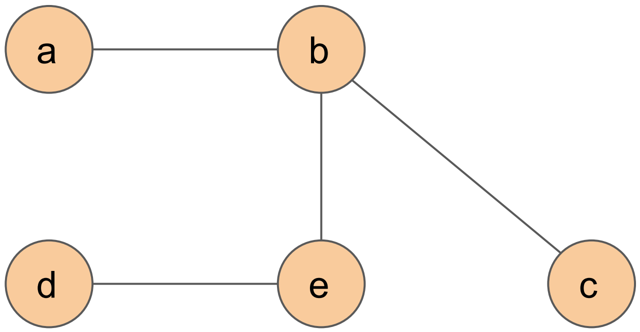


Example: diffusion of a message m *from process* e.

- e *issues* m.
- b *and* d *receive* m.
- *a and* c *receive* m.

Gossip is **correct** if and only if, if the sender is correct, every correct process eventually receives the message.

Exercise 4 (gossip)

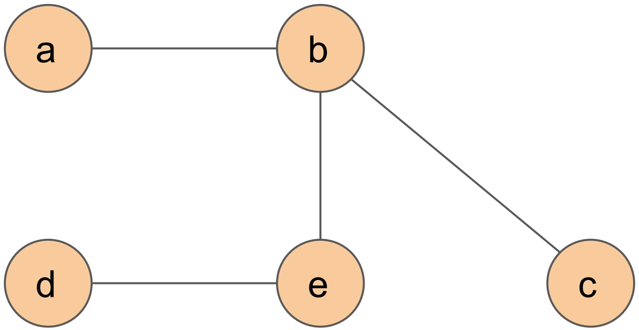
Prove that gossip is correct if and only if the subgraph of correct processes is connected.

Note: prove **both** directions of the implication!

Hint: induction is your friend.

Exercise 5 (gossip)

In the following system, exactly one process crashes. What is the minimum number of edges we need to add so that gossip is always correct?



k-connectivity

Two paths *p*, *p'* connecting two vertices *a* and *z* are **disjoint** if they have no vertex in common, except *a* and *z*:

$$p = (a, b, \ldots, y, z)$$

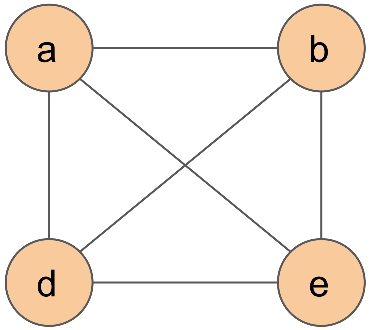
$$p' = (a, b', \ldots, y', z)$$

$$\{a, b, \ldots, y, z\} \cap \{a, b', \ldots, y', z\} = \{a, z\}$$

A graph is ***k*-connected** if and only if *k* disjoint paths exist between any two vertices of the graph.

Robustness

Gossip is **robust** to *k* failures if and only if it is always correct, as long as no more than *k* nodes are crashed.



A fully connected gossip graph is robust to N failures, where N is the number of processes.

Exercise 6 (robustness)

Prove that, if the gossip graph is *(k+1)*-connected, then gossip is *k*-robust.

Is the converse also true? Find a counterexample if not.

Hint: contradiction is your friend.

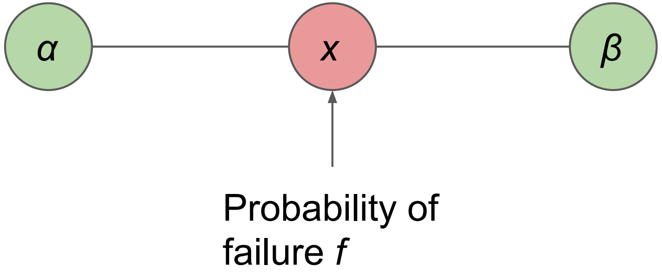
Random failures

Suppose that processes can fail independently with probability *f.*

What is the probability that two *correct* processes can communicate in the presence of failures?

It depends on their connectivity!

e.g.



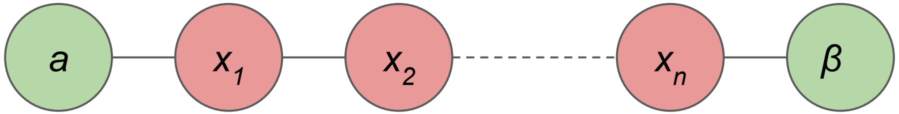
α, β can communicate iff *x* has not failed =>

α, β communicate with probability *1-f.*

Exercise 7 (random failures on series topology)

Suppose that processes x_i, $i=1, \ldots, n$ can fail independently with probability f.

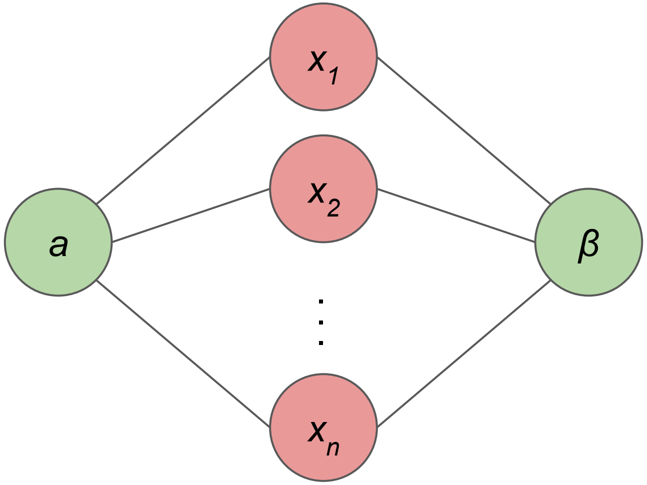
What is the probability that a and b can communicate?



Exercise 8 (random failures on parallel topology)

Suppose that processes x_i, $i=1, \ldots, n$ can fail independently with probability f.

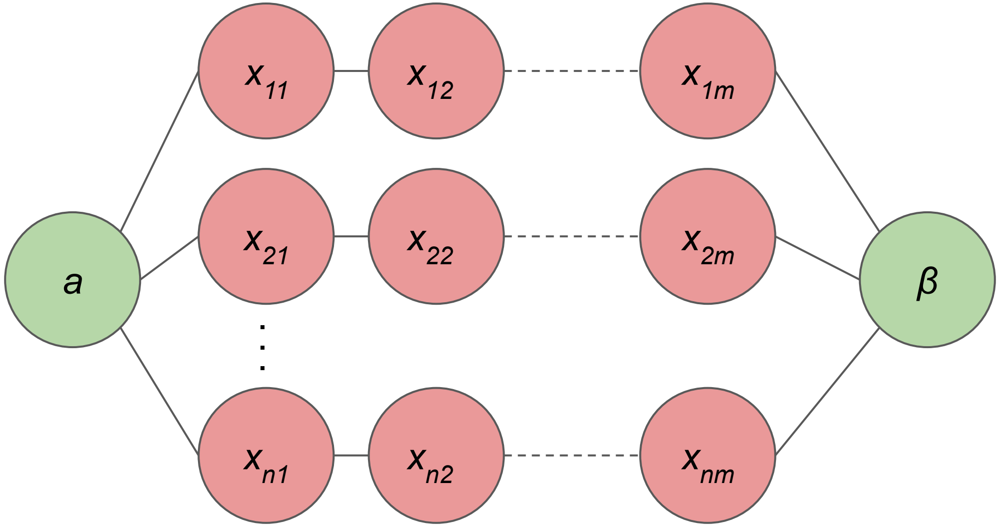
What is the probability that *a* and *b* can communicate?



Exercise 9 (random failures on series/parallel topology)

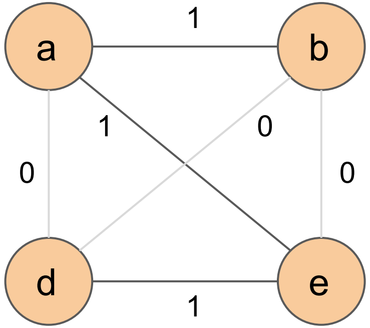
Suppose that processes x_{ij}, $i=1, \ldots, n$, $j=1, \ldots, m$ can fail independently with probability f.

Prove that a and b can communicate with probability $1 - [1 - (1-f)^m]^n$.



Erdös-Renyi graphs

An Erdös-Renyi graph *G(N, p)* is a random undirected graph with *N* vertices, such that any two distinct vertices have an independent probability *p* of being adjacent.



Example graph
G(4, ½)

An Erdös-Renyi graph is defined by the values of *N(N - 1)/2* independent Bernoulli random variables:

$$E_{ij} \sim Bernoulli(p)$$
$$E_{ij} = E_{ji}$$

with $i, j \in V$. Vertices *i* and *j* are adjacent iff $E_{ij} = 1$.

Bonus Exercise 10 (Erdös-Renyi graphs)

What distribution underlies the number of edges in an Erdös-Renyi *G(N, p)*?
What distribution underlies the degree (i.e., number of links) of any vertex?
Are the degrees of any two vertices independently distributed?

Hint: how is the sum of Bernoulli variables distributed?

Connectivity of *G(N, p)*

Let *C(N, p)* denote the probability of a random graph *G(N, p)* being connected.
It is possible to prove that:

$$\lim_{N\to\infty} G(N, p) = 0 \qquad \text{iff } p < \ln(N) / N$$
$$\lim_{N\to\infty} G(N, p) = 1 \qquad \text{iff } p > \ln(N) / N$$

A large Erdös-Renyi graph is almost surely connected, as long as each vertex has an expected degree larger than *ln(N)*.

We can use Erdös-Renyi graphs to build probabilistic gossip with *logarithmic communication complexity*!

Bonus Exercise 11 (Erdös-Renyi graphs)

Write a distributed procedure that runs on *N* processes to build an Erdös-Renyi graph *G(N, ln(N)).* We assume no failures. Each process can invoke:

- A procedure *rand(x)* that returns a real number between *0* and *x*, independently picked with uniform probability.
- A procedure *connect(i)* to connect to the *i*-th process.

Is it possible for the procedure to have *O(ln(N))* computation complexity?