Distributed Algorithms

Fall 2019

Logic 101
1st exercise session, 23/09/2019

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Example 1 (Conditional statements)

Write the converse, contrapositive and inverse of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:

Let P be the proposition \( p \rightarrow q \):

- The converse of P is: \( q \rightarrow p \):
- The inverse of P is: \( \neg p \rightarrow \neg q \)
- The contrapositive of P is: \( \neg q \rightarrow \neg p \)

Notes:

- Only the contrapositive of a conditional statement is equivalent to it.
- The proposition “p iff q” means that both P and the converse of P are true.
Exercise 1 (Conditional statements)

Write the negation of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:

Let P be a proposition. The negation of P is ¬P (“not P”). For example:

- ¬“7 is odd” = “7 is not odd” = “7 is even” (if you prove it!)
- ¬“All cats are animals” = “Some cats are not animals”

Hints:

- The negation of ¬p → ¬q is not p → ¬q.
- Express the implication in terms of and and or expressions.
Example 2

If the following statement is true:

*If process $i$ fails, then instantly all processes $j \neq i$ fail*

Which of the following are also true?

1. If a process $j \neq i$ fails, then process $i$ has failed,
2. If a process $j \neq i$ fails, nothing can be said about process $i$,
3. If a process $j \neq i$ fails, then process $i$ has not failed,

*(continues on next slide)*
Example 2 (contd)

4. If no process j≠i fails, nothing can be said about process i,
5. If no process j≠i fails, then process i has failed,
6. If no process j≠i fails, then process i has not failed,
7. If all processes j≠i fail, then process i has failed,
8. If all processes j≠i fail, nothing can be said about process i,
9. If all processes j≠i fail, then process i has not failed,
10. If some process j≠i does not fail, nothing can be said about process i,
11. If some process j≠i does not fail, then process i has failed,
12. If some process j≠i does not fail, then process i has not failed.
Exercise 2

Replace “instantly” with “eventually” in Example 2.
Example 3 (Proof by cases)

Let \(x, y, z, q\) be natural numbers such that

\[x^2 + 5y^2 + 5z^2 = q^2\]

Prove that \(q\) is even if and only if all \(x, y,\) and \(z\) are even as well.
Exercise 3 (Proof by cases)

Prove that $x + |x - 7| \geq 7$
Example 4 (Proof by contradiction)

Prove that the set of prime numbers is infinite.
Exercise 4 (Proof by contradiction)

Prove that if $\alpha^2$ is even, $\alpha$ is even.
Bonus Exercise 4 (Proof by contradiction)

Prove that $\sqrt{2}$ is irrational.

*Hint: Use the result of exercise 4*

**Proof by contradiction:**

- In order to prove $p$, find a contradiction $q$ such that $\neg p \rightarrow q$ is true.
- A contradiction always has the form: $q \equiv r \land \neg r$.

**Hints:**

- Use the result of Exercise 3.
- A rational number is always in the form $r/q$, where $r$ is integer, $q$ is natural, and $r$ and $q$ have *no common divisor*.
Example 5 (proof by induction)

Let a Swiss chocolate of rectangular shape $m \times n$. What is the smallest number of cuts we need to do, in order to break up the chocolate in individual pieces of size $1 \times 1$?

A cut is defined as any line that:

1. Does not cross itself,
2. Starts and ends on the perimeter of the chocolate piece it cuts.

*Note:* You cannot consider a cut on two pieces as a single cut, just because these pieces are next to each other.
Exercise 5 (proof by induction)

A chessboard of size $2^n \times 2^n$ ($n \geq 0$) has all of its squares painted white, except for one arbitrary square, which is painted black.

Prove that for every $n \geq 0$, you can cover all the white squares of the chessboard with L-shaped non-overlapping tiles.

e.g.  
$n = 0$  
$n = 1$  
$n = 2$  
L-shaped tile
Bonus Exercise 5 (proof by induction)

Consider a country with $n \geq 2$ cities. For every pair of different cities $x$, $y$, there exists a direct route (single direction) either from $x$ to $y$ or from $y$ to $x$. Show that there exists a city that we can reach from every other city either directly or through exactly one intermediate city.