Example 1 (Conditional statements)

Write the converse, contrapositive and inverse of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:
Let P be the proposition \( p \to q \):

- The converse of P is: \( q \to p \):
- The inverse of P is: \( \neg p \to \neg q \)
- The contrapositive of P is: \( \neg q \to \neg p \)

Notes:
- Only the contrapositive of a conditional statement is equivalent to it.
- The proposition “p iff q” means that both P and the converse of P are true.
Exercise 1 (Conditional statements)

Write the negation of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:

Let P be a proposition. The negation of P is ¬P (‘not P’). For example:

- ¬“7 is odd” = “7 is not odd” = “7 is even” (if you prove it!)
- ¬“All cats are animals” = “Some cats are not animals”

Hints:

- The negation of ¬p → ¬q is not p → ¬q.
- Express the implication in terms of and and or expressions.
Example 2

If the following statement is true:

*If process $i$ fails, then instantly all processes $j \neq i$ fail*

Which of the following are also true?

1. If a process $j \neq i$ fails, then process $i$ has failed,
2. If a process $j \neq i$ fails, nothing can be said about process $i$,
3. If a process $j \neq i$ fails, then process $i$ has not failed,

*(continues on next slide)*
Example 2 (contd)

4. If no process $j \neq i$ fails, nothing can be said about process $i$,
5. If no process $j \neq i$ fails, then process $i$ has failed,
6. If no process $j \neq i$ fails, then process $i$ has not failed,
7. If all processes $j \neq i$ fail, then process $i$ has failed,
8. If all processes $j \neq i$ fail, nothing can be said about process $i$,
9. If all processes $j \neq i$ fail, then process $i$ has not failed,
10. If some process $j \neq i$ does not fail, nothing can be said about process $i$,
11. If some process $j \neq i$ does not fail, then process $i$ has failed,
12. If some process $j \neq i$ does not fail, then process $i$ has not failed.
Exercise 2

Replace “instantly” with “eventually” in Example 2.
Exercise 2 (solution)

1. False: Some process $j$ can fail for a reason not related to the failure of process $i$.
2. True: explanation in (1).
3. False: explanation in (1).
4. True: Because of “eventually”.
5. False.
6. False: Because of “eventually”.
7. False.
8. True: Nothing can be said about process $i$.
10. True: Nothing can be said about process $i$, because of “eventually”.
11. False.
12. False: Nothing can be said about process $i$, because of “eventually”.

Example 3 (Proof by cases)

Let $x, y, z, q$ be natural numbers such that

$$x^2 + 5y^2 + 5z^2 = q^2$$

Prove that $q$ is even if and only if all of $x, y, \text{ and } z$ are even as well.
Exercise 3 (Proof by cases)

Prove that $x + |x - 7| \geq 7$
Exercise 3 (solution)

For the set of real numbers, we know that:

- $|a| = -a$, if $a < 0$
- $|a| = a$, if $a \geq 0$

So:

- If $x < 7$: $|x - 7| = 7 - x$, therefore $x + |x - 7| = x + (7 - x) = 7 \geq 7$
- If $x \geq 7$: $|x - 7| = x - 7$, therefore $x + |x - 7| = x + (x - 7) = 2x - 7 \geq 2 \cdot 7 - 7 \rightarrow x + |x - 7| \geq 7$
Example 4 (Proof by contradiction)

Prove that the set of prime numbers is infinite.
Exercise 4 (Proof by contradiction)

Prove that if $\alpha^2$ is even, $\alpha$ is even.
Exercise 4 (solution)

When we want to prove something by contradiction, we start by assuming that the negation (of whatever we are trying to prove) is true.

We said in the classroom that $p \rightarrow q$ is equivalent to $\neg p \lor q$. Therefore the negation of $p \rightarrow q$ is $p \land \neg q$.

With that said, let's assume that $a^2$ is even and $a$ is odd. Since $a$ is odd, $a$ can be written as $a=2k+1$. Therefore, $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Thus, $a^2$ is odd, a contradiction!
Bonus Exercise 4 (Proof by contradiction)

Prove that $\sqrt{2}$ is irrational.

*Hint: Use the result of exercise 4*

**Proof by contradiction:**

- In order to prove $p$, find a contradiction $q$ such that $\neg p \rightarrow q$ is true.
- A contradiction always has the form: $q \equiv r \land \neg r$.

**Hints:**

- Use the result of Exercise 3.
- A rational number is always in the form $r/q$, where $r$ is integer, $q$ is natural, and $r$ and $q$ have no common divisor.
Exercise 4 (Solution)

- Assume that $\sqrt{2}$ is rational, i.e. $\sqrt{2} = a/b$ where $a,b$ are coprime (have no common divisors).
- We square both sides, thus $2 = a^2/b^2 \rightarrow a^2 = 2b^2$.
- Therefore, $a^2$ is even, and using the result of the previous exercise we know that $a$ is even.
- Since $a$ is even, it has the form $a=2k$. We substitute this in the previous equation and we have that:
  - $(2k)^2 = 2b^2 \rightarrow 4k^2 = 2b^2 \rightarrow b^2 = 2k^2$.
  - Since $b^2 = 2k^2$, this means that $b^2$ is even $\rightarrow b$ is even, which is a contradiction!
- The contradiction is that we assumed $a,b$ to be coprime, but we concluded that both are even!
Example 5 (proof by induction)

Let a Swiss chocolate of rectangular shape $mxn$. What is the smallest number of cuts we need to do, in order to break up the chocolate in individual pieces of size 1x1?

A cut is defined as any line that:

1. Does not cross itself,
2. Starts and ends on the perimeter of the chocolate piece it cuts.

Note: You cannot consider a cut on two pieces as a single cut, just because these pieces are next to each other.
Exercise 5 (proof by induction)

A chessboard of size $2^n \times 2^n$ ($n \geq 0$) has all of its squares painted white, except for one arbitrary square, which is painted black.

Prove that for every $n \geq 0$, you can cover all the white squares of the chessboard with L-shaped non-overlapping tiles.

e.g.

\begin{align*}
\text{} & \quad n = 0 \\
\begin{array}{c}
\text{} \\
\end{array} & \quad n = 1 \\
\begin{array}{c}
\text{} \\
\end{array} & \quad n = 2 \\
\end{align*}

L-shaped tile
Exercise 5 (solution 1/2)

We will use induction:

- Base case (n=0): We can tile one black square, using 0 L-shaped tiles.
- Inductive step: Suppose this property holds for \( n \geq 0 \):
  - i.e., we can tile a \( 2^n \times 2^n \) grid using L-shaped tiles, leaving a single square uncovered (the black square) at an arbitrary location. We will show how to tile a \( 2^{n+1} \times 2^{n+1} \) grid.
Suppose the grid has size $2^{n+1} \times 2^{n+1}$ (we show a grid for $n=3$) and the black square is somewhere in the grid.

We split the $2^{n+1} \times 2^{n+1}$ grid in 4 sub-grids of size $2^n \times 2^n$.

We can tile each sub-grid because of the inductive step. For the top-left sub-grid we leave the green square uncovered. We also leave the blue and the red squares uncovered in their corresponding sub-grids.

For the three squares in the middle, we can use an L-shaped tile.
Bonus Exercise 5 (proof by induction)

Consider a country with \( n \geq 2 \) cities. For every pair of different cities \( x, y \), there exists a direct route (single direction) either from \( x \) to \( y \) or from \( y \) to \( x \). Show that there exists a city that we can reach from every other city either directly or through exactly one intermediate city.
Exercise 5 (solution 1/2)

We name “central” the a city that we can reach from every other city either directly or through exactly one intermediate city.

Base case (n=2): It obviously holds. Either one of the cities is “central”.

Inductive step: Suppose this property holds for $n \geq 2$ cities. We will prove that it will still hold for $n+1$ cities.
Let $n+1$ cities, $c_i$, $i=0, \ldots, n$, where for every pair of different cities $c_i, c_j$, there exists a direct route (single direction) either from $c_i$ to $c_j$ or from $c_j$ to $c_i$.

We consider only the first $n$ cities, i.e. cities $c_i$, $i=0, \ldots, n-1$. According to the inductive step, there exists one central city among these $n$ cities. Let $c_j$ be that city.

We now exclude city $c_j$ and consider the rest of the cities. Again, we have $n$ cities, therefore there should exist one city among them that is central. Let $c_k$ be that city.

All cities apart from $c_j$ and $c_k$ can reach $c_j$ and $c_k$ either directly or through one intermediate city.

Furthermore, there exists a route between $c_j$ and $c_k$:
- If the route is directed from $c_j$ to $c_k$, then $c_k$ is the central city for the $n+1$ cities.
- If the route is directed from $c_k$ to $c_j$, then $c_j$ is the central city for the $n+1$ cities.