Distributed Algorithms
Fall 2020

Logic 101
1st exercise session, 28/09/2020

Matteo Monti <matteo.monti@epfl.ch>
Jovan Komatovic <jovan.komatovic@epfl.ch>
Example 1 (Conditional statements)

Write the converse, contrapositive and inverse of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:

Let P be the proposition $p \rightarrow q$:

- The converse of P is: $q \rightarrow p$:
- The inverse of P is: $\neg p \rightarrow \neg q$
- The contrapositive of P is: $\neg q \rightarrow \neg p$

Notes:

- Only the contrapositive of a conditional statement is equivalent to it.
- The proposition “p iff q” means that both P and the converse of P are true.
Exercise 1 (Conditional statements)

Write the negation of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:
Let P be a proposition. The negation of P is ¬P (“not P”). For example:

- ¬“7 is odd” = “7 is not odd” = “7 is even” (if you prove it!)
- ¬“All cats are animals” = “Some cats are not animals”

Hints:
- The negation of ¬p → ¬q is not p → ¬q.
- Express the implication in terms of and and or expressions.
Example 2

If the following statement is true:

\textit{If process i fails, then instantly all processes }j \neq i \textit{ fail}

Which of the following are also true?

1. If a process }j \neq i \textit{ fails, then process }i \textit{ has failed,
2. If a process }j \neq i \textit{ fails, nothing can be said about process }i,
3. If a process }j \neq i \textit{ fails, then process }i \textit{ has not failed

(continues on next slide)
Example 2 (contd)

4. If no process \( j \neq i \) fails, nothing can be said about process \( i \),
5. If no process \( j \neq i \) fails, then process \( i \) has failed,
6. If no process \( j \neq i \) fails, then process \( i \) has not failed,
7. If all processes \( j \neq i \) fail, then process \( i \) has failed,
8. If all processes \( j \neq i \) fail, nothing can be said about process \( i \),
9. If all processes \( j \neq i \) fail, then process \( i \) has not failed,
10. If some process \( j \neq i \) does not fail, nothing can be said about process \( i \),
11. If some process \( j \neq i \) does not fail, then process \( i \) has failed,
12. If some process \( j \neq i \) does not fail, then process \( i \) has not failed.
Exercise 2

Replace “instantly” with “eventually” in Example 2.
Example 3 (Proof by cases)

Let $x, y, z, q$ be natural numbers such that

$$x^2 + 5y^2 + 5z^2 = q^2$$

Prove that $q$ is even if and only if all of $x, y, \text{ and } z$ are even as well.
Exercise 3 (Proof by cases)

Prove that $x + |x - 7| \geq 7$
Example 4 (Proof by contradiction)

Prove that the set of prime numbers is infinite.
Exercise 4 (Proof by contradiction)

Prove that if $\alpha^2$ is even, $\alpha$ is even.
Bonus Exercise 4 (Proof by contradiction)

Prove that $\sqrt{2}$ is irrational.

*Hint: Use the result of exercise 4*

Proof by contradiction:

- In order to prove $p$, find a contradiction $q$ such that $\neg p \rightarrow q$ is true.
- A contradiction always has the form: $q \equiv r \land \neg r$.

Hints:

- Use the result of Exercise 3.
- A rational number is always in the form $r/q$, where $r$ is integer, $q$ is natural, and $r$ and $q$ have no common divisor.
Example 5 (proof by induction)

Let a swiss chocolate of rectangular shape $m \times n$. What is the smallest number of cuts we need to do, in order to break up the chocolate in individual pieces of size $1 \times 1$?

A cut is defined as any line that:

1. Does not cross itself,
2. Starts and ends on the perimeter of the chocolate piece it cuts.

*Note:* You cannot consider a cut on two pieces as a single cut, just because these pieces are next to each other.

![Diagram of the chocolate and cuts](image)
Exercise 5 (proof by induction)

A chessboard of size $2^n \times 2^n$ ($n \geq 0$) has all of its squares painted white, except for one arbitrary square, which is painted black.

Prove that for every $n \geq 0$, you can cover all the white squares of the chessboard with L-shaped non-overlapping tiles.

<table>
<thead>
<tr>
<th>n = 0</th>
<th>n = 1</th>
<th>n = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Black square]</td>
<td>![White squares with black square]</td>
<td>![White squares with black square]</td>
</tr>
</tbody>
</table>

L-shaped tile
Bonus Exercise 5 (proof by induction)

Consider a country with \( n \geq 2 \) cities. For every pair of different cities \( x, y \), there exists a direct route (single direction) either from \( x \) to \( y \) or from \( y \) to \( x \). Show that there exists a city that we can reach from every other city either directly or through exactly one intermediate city.