

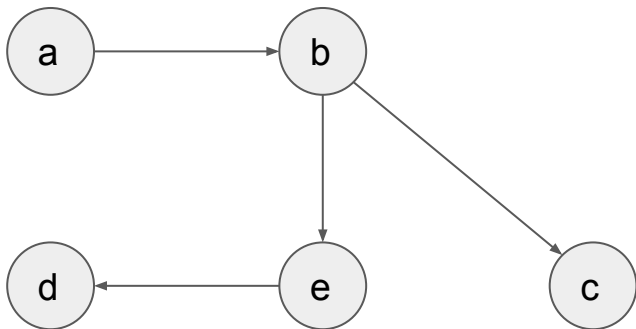
# Distributed Algorithms

Links & Gossip  
1st exercise session

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# Graphs

A **graph** is a couple  $(V, E)$  where  $V$  is a set of **vertices** and  $E \subseteq V^2$  is a set of **edges**.



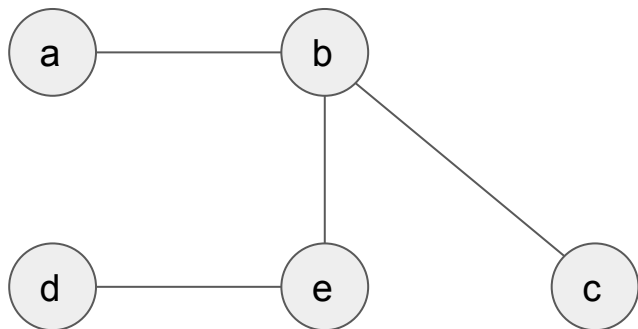
*Example graph*  $(V, E)$ :

- $V = \{a, b, c, d, e\}$
- $E = \{(a, b), (b, c), (b, e), (e, d)\}$

Two vertices are **adjacent** (or **neighbors**) iff an edge exists between them. In the example,  $a$  and  $b$  are adjacent;  $a$  and  $d$  are not adjacent.

# Graphs (undirected)

An **undirected graph** is a graph  $(V, E)$  such that  $(a, b) \in E$  if and only if  $(b, a) \in E$ .



*Example graph*  $(V, E)$ :

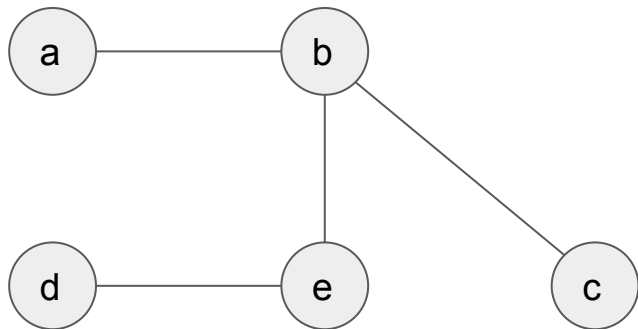
- $V = \{a, b, c, d, e\}$
- $E = \{(a, b), (b, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e)\}$

We use undirected graphs to model networks of processes:

- Each vertex represents a process
- Two vertices are neighbors iff the corresponding processes can directly exchange messages.

# Paths

A **path** is a sequence of *distinct* vertices  $(v_1, \dots, v_N)$  such that, for all  $i \in [1, N - 1]$ ,  $v_i$  and  $v_{i+1}$  are adjacent.



*Some paths in  $(V, E)$ :*

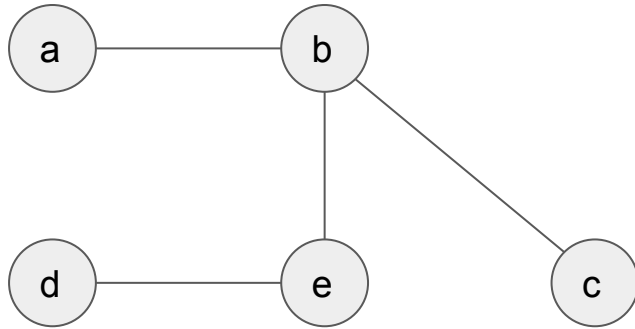
- $(a, b)$
- $(a, b, c)$
- $(a, b, e, d)$

*While*

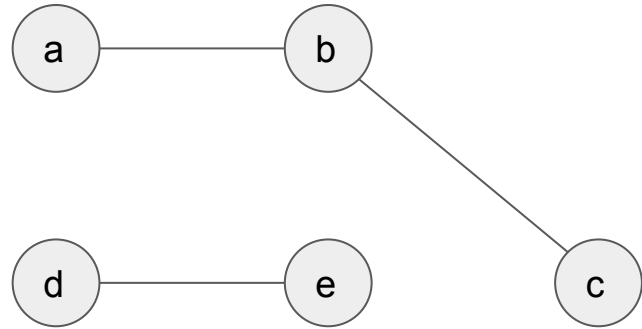
- $(a, c, e)$  is **not** a path: *a and c are not adjacent!*

# Connectivity

Two distinct vertices  $a$  and  $z$  are **connected** if and only if at least one path  $(a, \dots, z)$  exists in the graph. A graph is connected if any two distinct vertices are connected.



*A connected graph*



*A disconnected graph*

# Exercise 1 (connectivity)

Prove that **connectivity** is a **symmetric property** on an undirected graph: let  $a, b$  be vertices such that  $a$  is connected with  $b$ . Prove that  $b$  is connected with  $a$ .

*Hint: you can do it constructively.*

## Exercise 2 (connectivity)

Prove that **connectivity** is a **transitive property** on an undirected graph: let  $a$ ,  $b$ ,  $c$  be vertices such that  $a$  is connected with  $b$  and  $b$  is connected with  $c$ . Prove that  $a$  is connected with  $c$ .

*Hint: double-check the definition of a path.*

## Exercise 3 (connectivity)

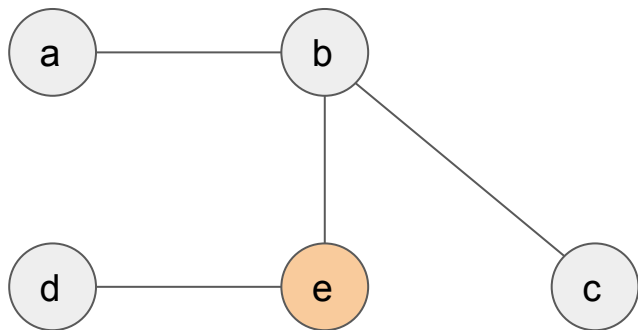
Write a procedure (pseudocode or any programming language) that inputs an undirected graph  $G = (V, E)$  and outputs *true* if and only if the  $G$  is connected.

*Hint: use the results from Exercises 1 and 2.*



# Gossip

We use an undirected graph to represent which processes can communicate. Upon receiving a new message  $m$ , a process forwards  $m$  to all its neighbors.

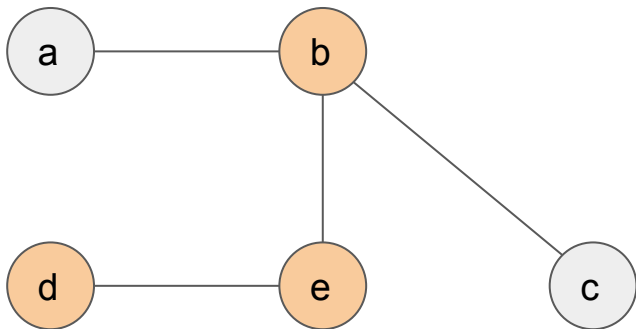


*Example: diffusion of a message  $m$  from process  $e$ .*

- *$e$  issues  $m$*

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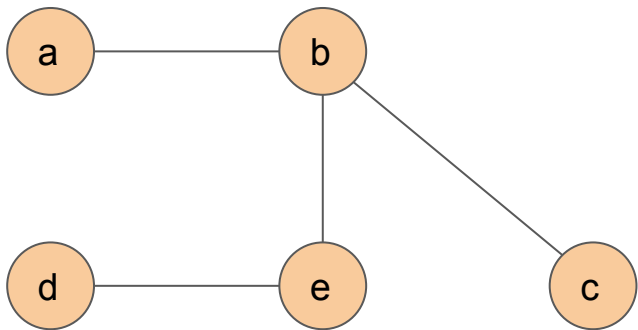


*Example: diffusion of a message  $m$  from process  $e$ .*

- *$e$  issues  $m$ .*
- *$b$  and  $d$  receive  $m$ .*

# Gossip

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*Example: diffusion of a message  $m$  from process  $e$ .*

- *$e$  issues  $m$ .*
- *$b$  and  $d$  receive  $m$ .*
- *$a$  and  $c$  receive  $m$ .*

Gossip is **correct** if and only if, if the sender is correct, every correct process eventually receives the message.

## Exercise 4 (gossip)

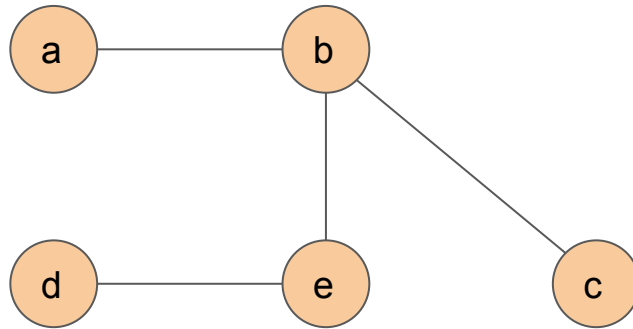
Prove that gossip is correct if and only if the subgraph of correct processes is connected.

*Note:* prove **both** directions of the implication!

*Hint: induction is your friend.*

## Exercise 5 (gossip)

In the following system, exactly one process crashes. What is the minimum number of edges we need to add so that gossip is always correct?



# $k$ -connectivity

Two paths  $p, p'$  connecting two vertices  $a$  and  $z$  are **disjoint** if they have no vertex in common, except  $a$  and  $z$ :

$$p = (a, b, \dots, y, z)$$

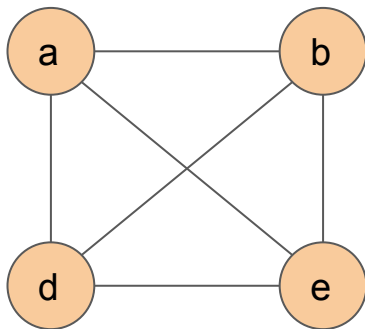
$$p' = (a, b', \dots, y', z)$$

$$\{a, b, \dots, y, z\} \cap \{a, b', \dots, y', z\} = \{a, z\}$$

A graph is  **$k$ -connected** if and only if  $k$  disjoint paths exist between any two vertices of the graph.

# Robustness

Gossip is **robust** to  $k$  failures if and only if it is always correct, as long as no more than  $k$  nodes are crashed.



*A fully connected gossip graph is robust to  $N$  failures, where  $N$  is the number of processes.*

## Exercise 6 (robustness)

Prove that, if the gossip graph is  $(k+1)$ -connected, then gossip is  $k$ -robust.

Is the converse also true? Find a counterexample if not.

*Hint: contradiction is your friend.*



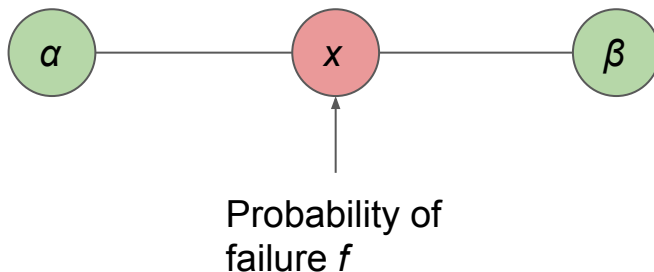
# Random failures

Suppose that processes can fail independently with probability  $f$ .

What is the probability that two *correct* processes can communicate in the presence of failures?

*It depends on their connectivity!*

e.g.



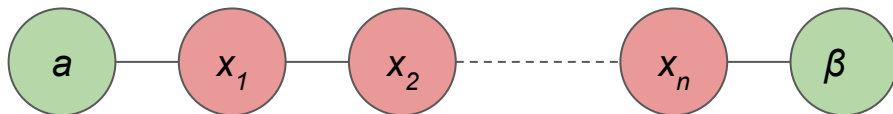
$\alpha, \beta$  can communicate iff  $x$  has not failed  $\Rightarrow$

$\alpha, \beta$  communicate with probability  $1-f$ .

## Exercise 7 (random failures on series topology)

Suppose that processes  $x_i$ ,  $i=1, \dots, n$  can fail independently with probability  $f$ .

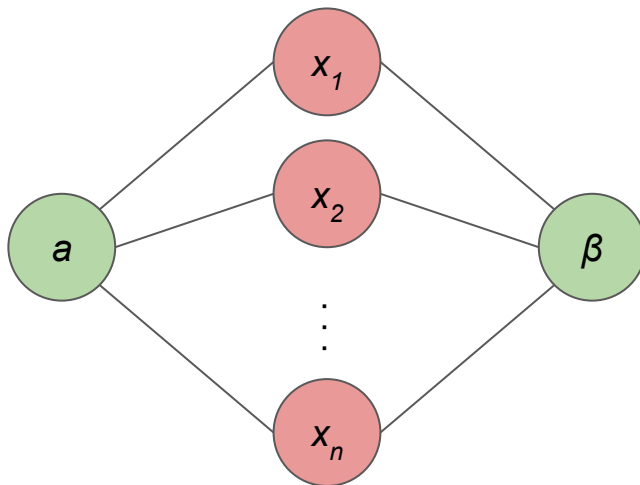
What is the probability that  $a$  and  $b$  can communicate?



## Exercise 8 (random failures on parallel topology)

Suppose that processes  $x_i$ ,  $i=1, \dots, n$  can fail independently with probability  $f$ .

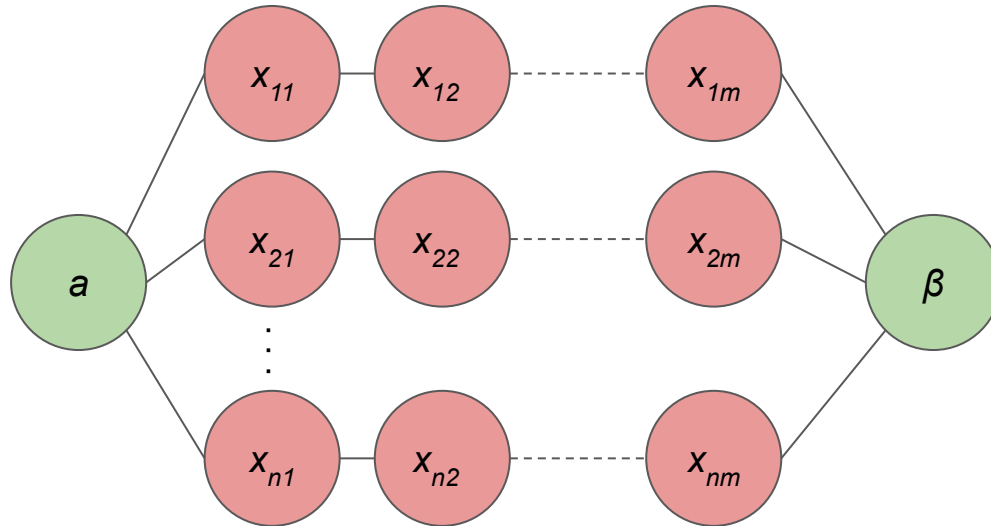
What is the probability that  $a$  and  $b$  can communicate?



# Exercise 9 (random failures on series/parallel topology)

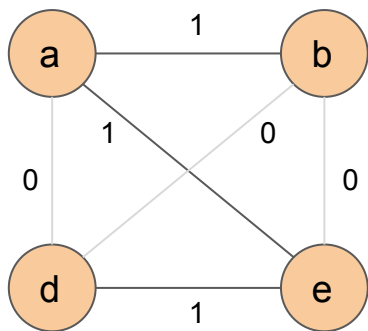
Suppose that processes  $x_{ij}$ ,  $i=1, \dots, n$ ,  $j=1, \dots, m$  can fail independently with probability  $f$ .

Prove that  $a$  and  $b$  can communicate with probability  $1 - [1 - (1-f)^m]^n$ .



# Erdős-Renyi graphs

An Erdős-Renyi graph  $G(N, p)$  is a random undirected graph with  $N$  vertices, such that any two distinct vertices have an independent probability  $p$  of being adjacent.



Example graph  
 $G(4, \frac{1}{2})$

An Erdős-Renyi graph is defined by the values of  $N(N - 1)/2$  independent Bernoulli random variables:

$$E_{ij} \sim \text{Bernoulli}(p)$$
$$E_{ij} = E_{ji}$$

with  $i, j \in V$ . Vertices  $i$  and  $j$  are adjacent iff  $E_{ij} = 1$ .

# Bonus Exercise 10 (Erdős-Renyi graphs)

What distribution underlies the number of edges in an Erdős-Renyi  $G(N, p)$ ?

What distribution underlies the degree (i.e., number of links) of any vertex?

Are the degrees of any two vertices independently distributed?

*Hint: how is the sum of Bernoulli variables distributed?*

# Connectivity of $G(N, p)$

Let  $C(N, p)$  denote the probability of a random graph  $G(N, p)$  being connected. It is possible to prove that:

$$\begin{aligned} \lim_{N \rightarrow \infty} C(N, p) &= 0 && \text{iff } p < \ln(N) / N \\ \lim_{N \rightarrow \infty} C(N, p) &= 1 && \text{iff } p > \ln(N) / N \end{aligned}$$

A large Erdős-Renyi graph is almost surely connected, as long as each vertex has an expected degree larger than  $\ln(N)$ .

We can use Erdős-Renyi graphs to build probabilistic gossip with  
*logarithmic communication complexity!*

# Bonus Exercise 11 (Erdős-Renyi graphs)

Write a distributed procedure that runs on  $N$  processes to build an Erdős-Renyi graph  $G(N, \ln(N)/N)$ . We assume no failures. Each process can invoke:

- A procedure  $rand(x)$  that returns a real number between 0 and  $x$ , independently picked with uniform probability.
- A procedure  $connect(i)$  to connect to the  $i$ -th process.

Is it possible for the procedure to have  $O(\ln(N))$  computation complexity?