Self-stabilizing edge coloring

We have previously seen the "node coloring problem": all nodes of a graph must eventually have a color different from their neighbors.

We now consider a slightly more difficult problem: the "edge coloring problem", where each edge (a link between two nodes) must eventually have a color different from the adjacent edges.

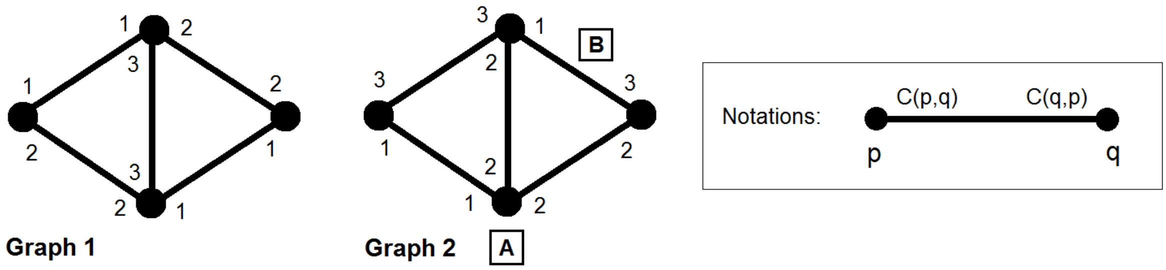
More formally, we consider a graph of degree D (at most D neighbors per node), and $2D-1$ colors. Each node is eventually activated, but two neighbor nodes are never activated at the same time.

For each node p, let $N(p)$ be the set of neighbors of p. $\forall q \in N(p)$, p has a variable $C(p,q)$, which corresponds to the color that p attributes to the edge $\{p,q\}$.

We say that the graph is "well colored" if the two following properties are satisfied:

- **Property 1**: For each node p, for any two neighbors q and r of p, $C(p,q) \neq C(p,r)$ (that is, the edges connected to a same node must have distinct colors).
- **Property 2**: For each edge $\{p,q\}$, $C(p,q) = C(q,p)$ (that is, p and q attribute the same color to $\{p,q\}$).

This is illustrated in the following figure. Graph 1 is "well colored": each node satisfies property 1 and each edge satisfies property 2. Graph 2 is "not well colored": for instance, the node A does not satisfy property 1, and the edge B does not satisfy property 2.



We consider the following edge coloring algorithm:

When a node p is activated...

1. If p does not satisfy property 1, attribute any distinct colors to the edges of p.
2. If p satisfies property 1, for each neighbor q of p such that $C(p,q) \neq C(q,p)$:
   Let C_p (resp. C_q) be the set of colors that p (resp. q) attributes to all its edges except $\{p,q\}$.
   (a) If $C(q,p) \notin C_p$, then $C(p,q) := C(q,p)$
   (b) If $C(q,p) \in C_p$, let $S = \{1,\ldots,2D-1\} - C_p - C_q$ (the set of colors neither in C_p nor C_p). If S is not empty, let $C \in S$. Then, $C(p,q) := C$.

Questions

1. Show that the set S of the algorithm always contains at least one element.
2. Show that each node p eventually satisfies property 1, and then always keeps satisfying this property. You can take inspiration from the liveness and safety proofs of the node coloring algorithm.
3. Suppose that all nodes satisfy property 1. Show that each edge $\{p,q\}$ eventually satisfies property 2, and always keeps satisfying this property. Then, conclude.