Consensus with Byzantine failures and asynchrony: the Ben-Or's algorithm, revisited

Distributed Algorithms

Recap

Positive Result: *There exists* a deterministic synchronous protocol that solves consensus, while tolerating crash failures

FLP Theorem: **No deterministic** protocol can solve consensus, while tolerating **1 crash** and **asynchrony**

Today

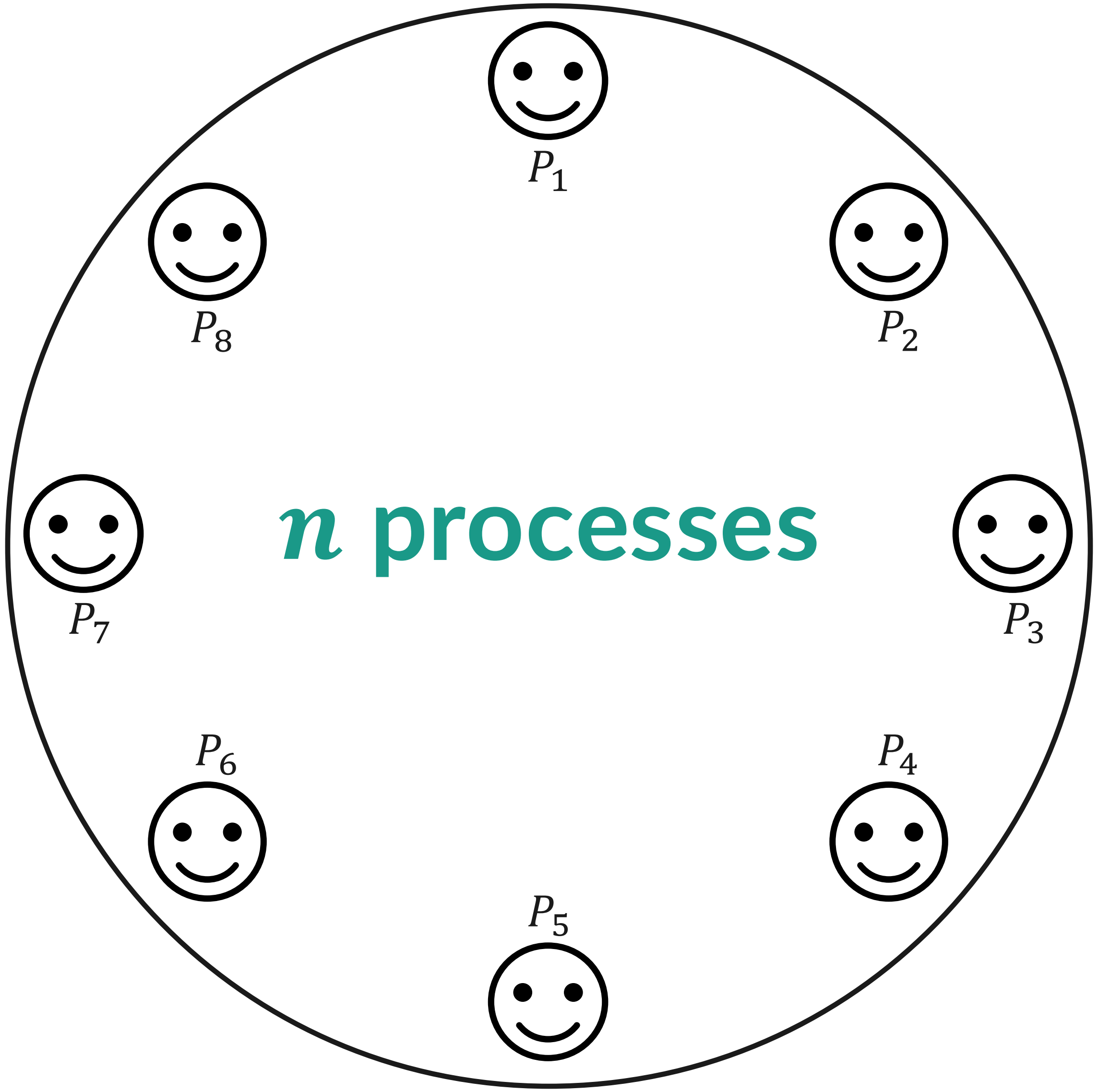
Positive Result 1: *There exists* a **randomized asynchronous** protocol that solves consensus, while tolerating arbitrary (**Byzantine**) failures

Positive Result 2: *There exists* a **deterministic synchronous** protocol that solves consensus, while tolerating arbitrary (**Byzantine**) failures

Consensus

Problem definition

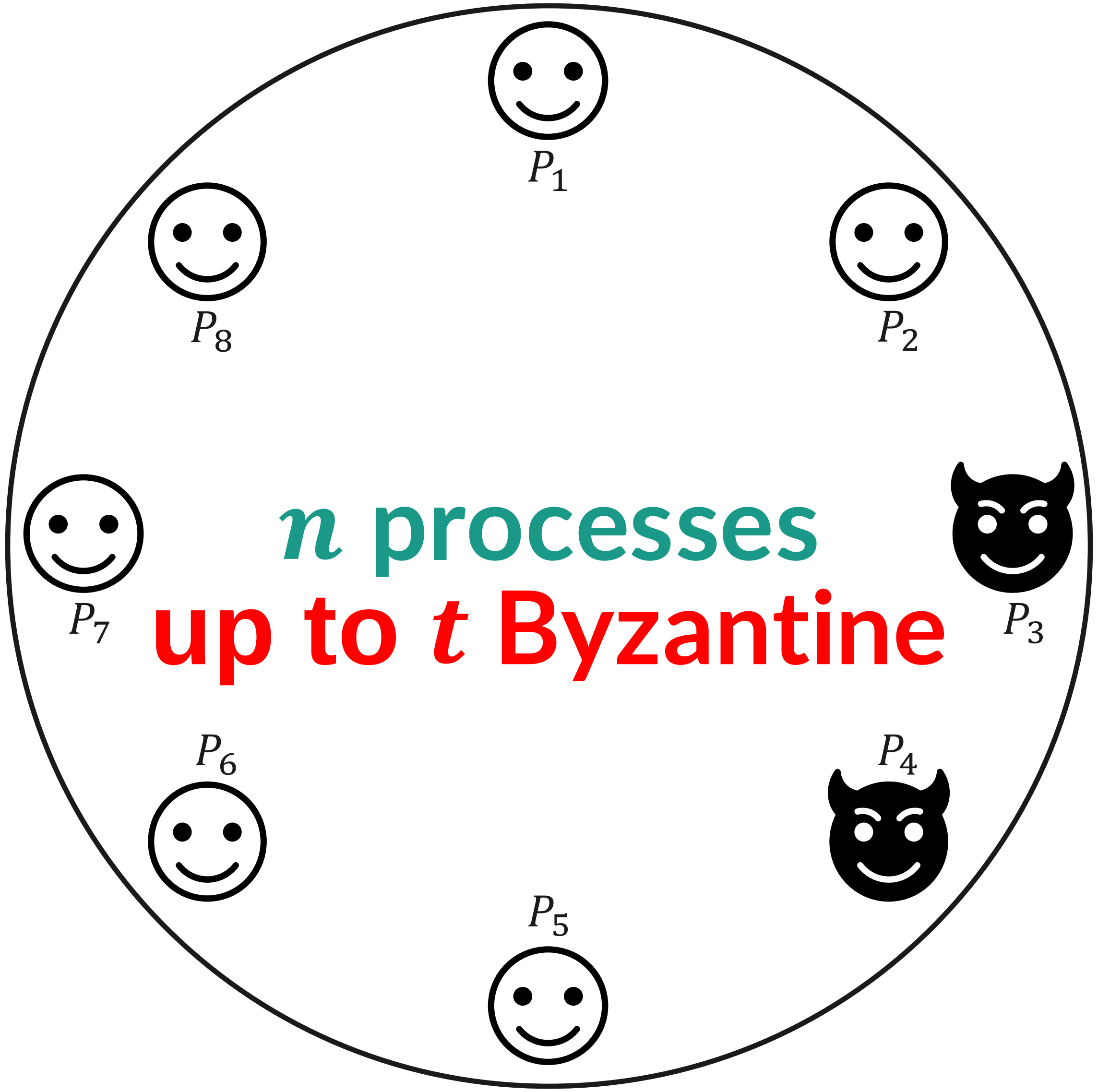
System



Static set of n publicly known identities

Message passing through reliable point-to-point channels

System



Static set of n publicly known identities

Message passing through reliable authenticated point-to-point channels

Interface of Consensus

Propose

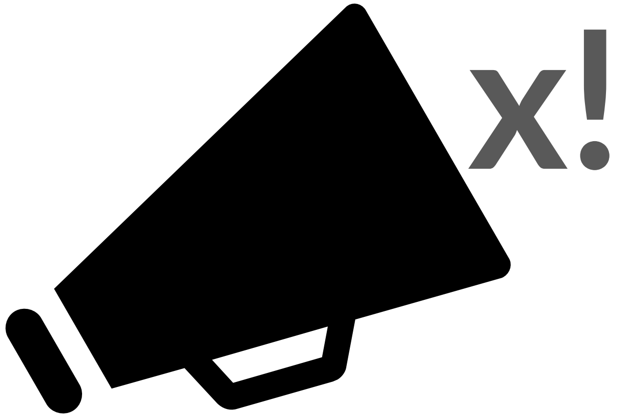
Operation

Decide

Callback

Interface of Consensus

Propose
Operation



Decide
Callback

Interface of Consensus

Propose

Operation

x!

Decide

Callback

y!

Properties of Consensus

- **Termination:** All correct processes decide.
- **Integrity:** No process decides more than once.
- **Agreement:** No two correct processes decide different values.
- **Validity:** A decided value is proposed by a correct process.

Asynchronous Model

No notion of time

Asynchronous Model (Informal)

- No shared global clock: no shared notion of time
- Arbitrary (but finite) message delays
- Model is purely event-driven: reception → sending

Synchronous Model

Known upper bound Δ on message delays
→ rounds of the form compute, send, receive

Graded Consensus

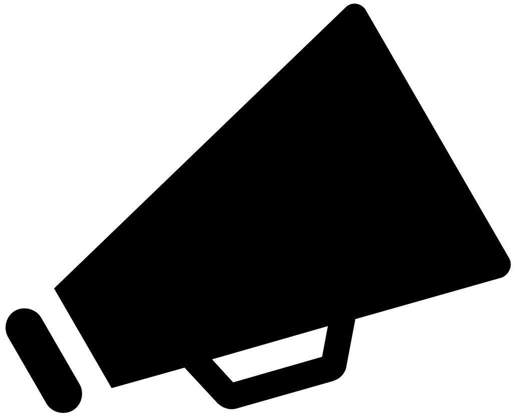
« Stay safe »

Graded Consensus

Specification

Interface of graded consensus

Propose
Operation



Decide
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Interface of graded consensus

Propose
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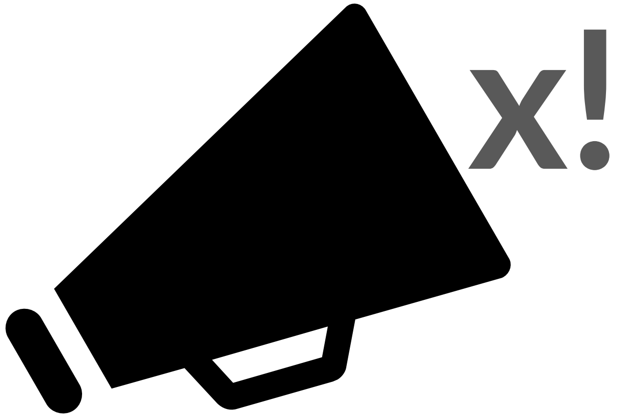

Decide
Callback

Interface of graded consensus

Propose
Operation

x!

Decide
Callback

(y,g)!

Interface of graded consensus

Propose
Operation

x!

Decide
Callback

(y,g)!

Grade = ‘confidence’

$g_{min}, \dots, g_{max}$

Properties of Graded Consensus

- **Termination:** All correct processes decide.
- **Integrity:** No process decides more than once.
- **Unanimity:** If only v is proposed, then only [redacted] can be decided.
- **Consistency (~~Agreement~~):** [redacted]

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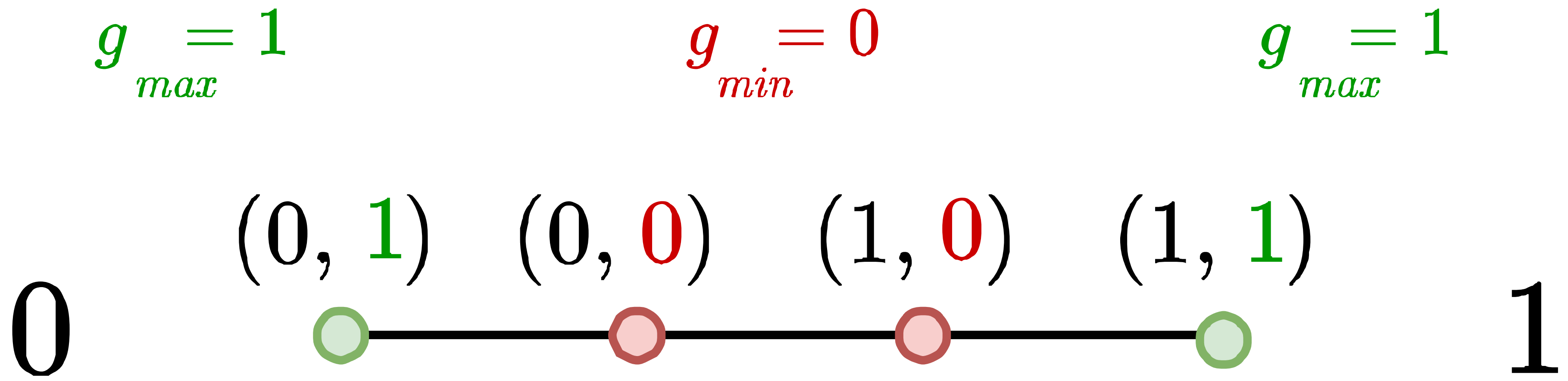
$g_{max} = 1$ $\quad$ $g_{min} = 0$ $\quad$ $g_{max} = 1$

$(0, 1)$ $\quad$ $(0, 0)$ $\quad$ $(1, 0)$ $\quad$ $(1, 1)$

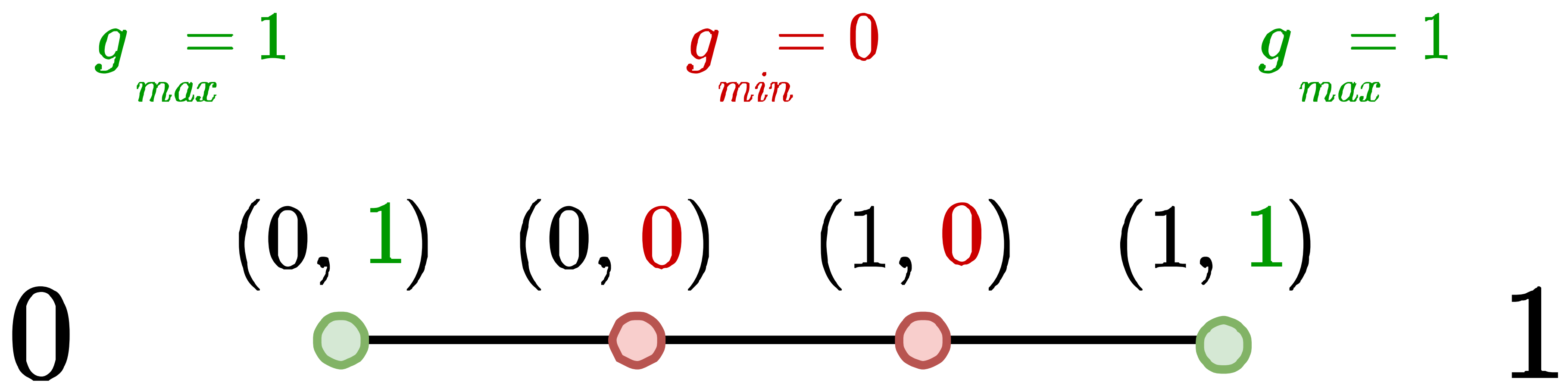
0

1

Consistency + Termination + Unanimity



Consistency + Termination + Unanimity



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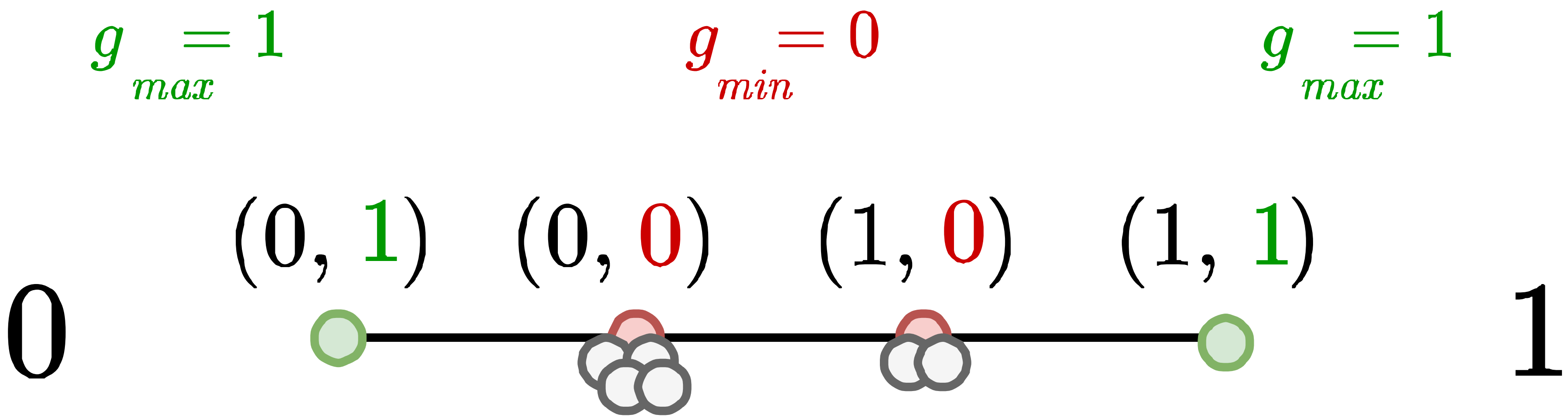

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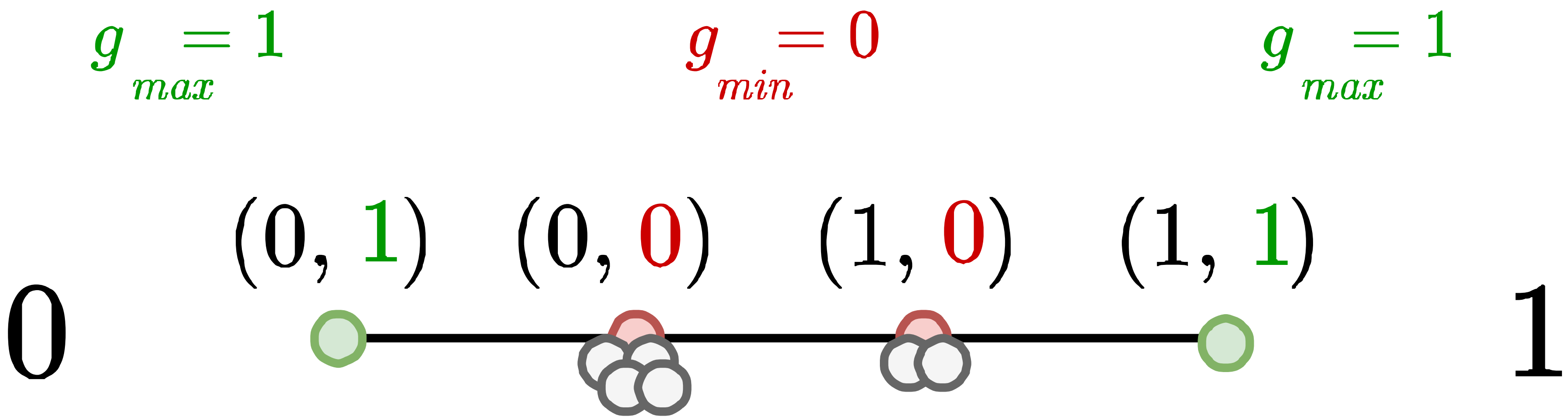
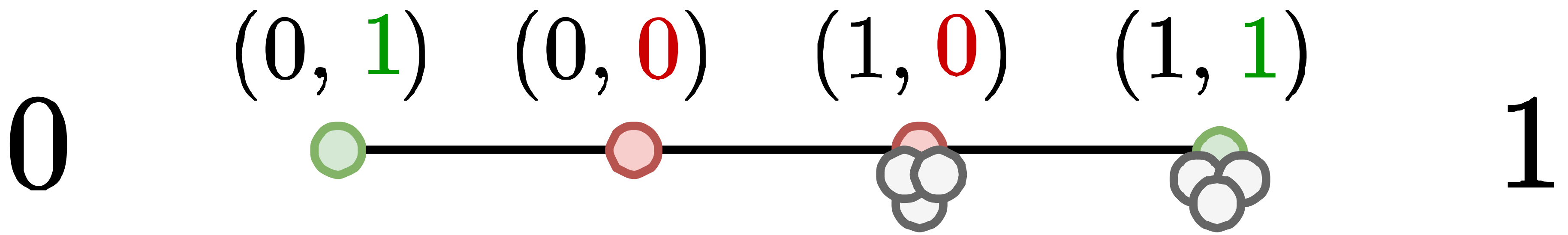


(1) If two correct processes decide (v,g) and (v',g'), then **either** (i) $\boxed{g=g'=0}$, **or** (ii) $|g-g'| \leq 1$, and v=v'.

(2) If two correct processes decide (v,g) and (v',g'), then (a) $|g-g'| \leq 1$, **and** (b) ($v \neq v' \implies g=g'=0$).

Consistency + **Termination** + **Unanimity**

$g_{max} = 1$ $g_{min} = 0$ $g_{max} = 1$



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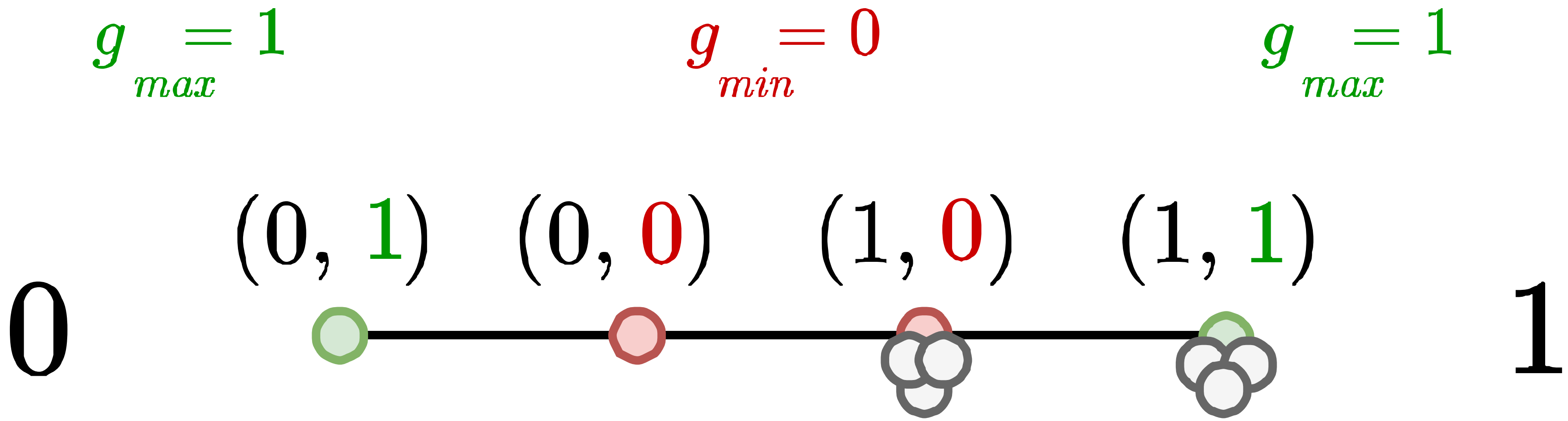

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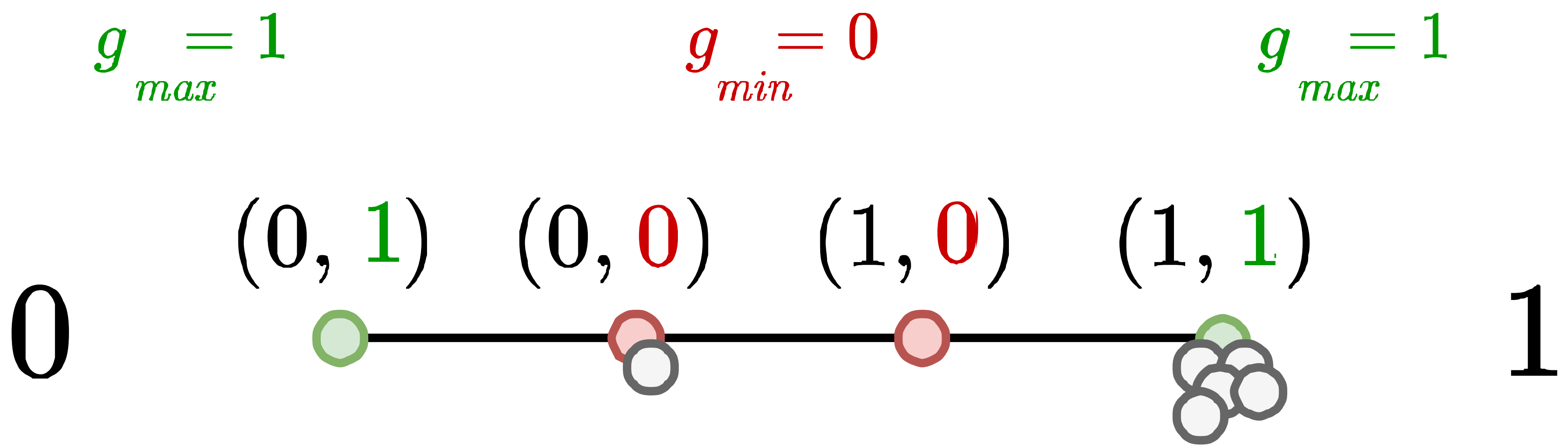

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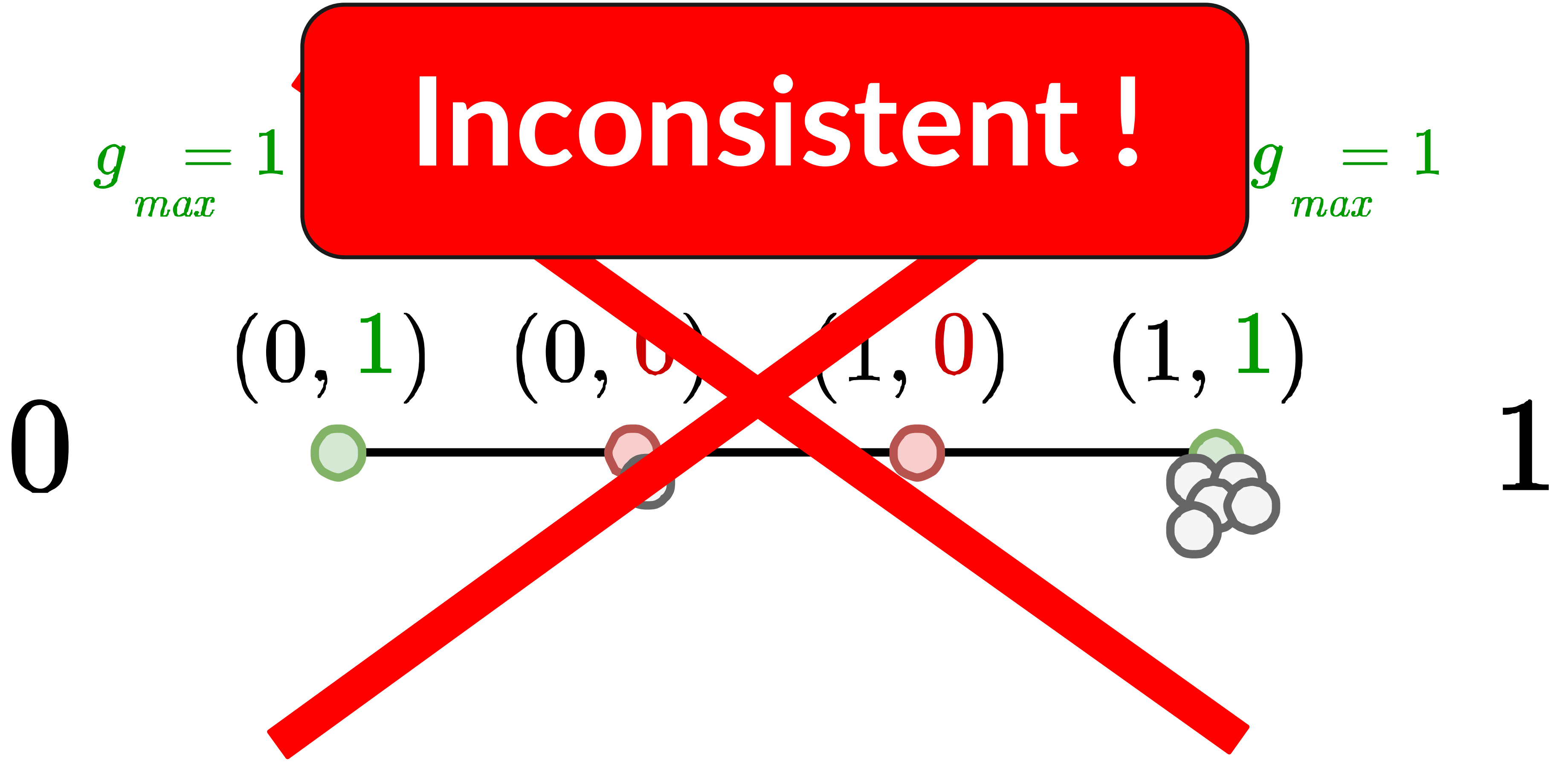

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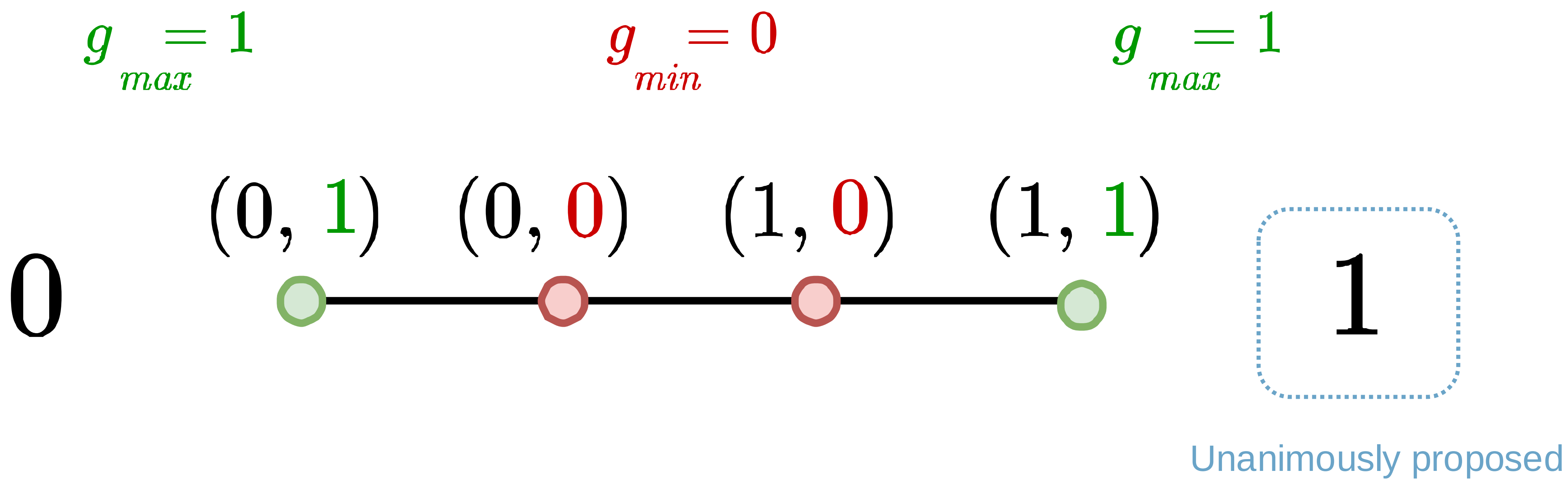
Consistency + Termination + Unanimity



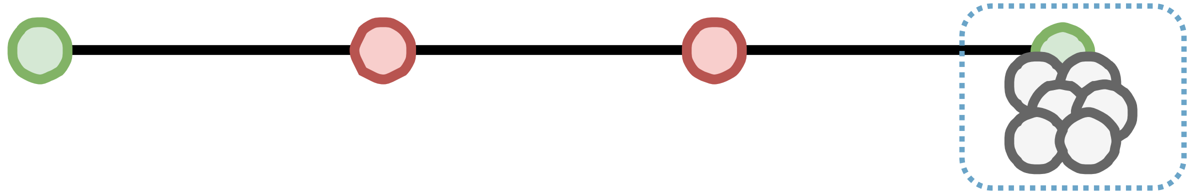
All correct processes eventually decide the unanimous proposal with high confidence value.

Consistency + Termination + Unanimity

$g_{max} = 1$ $g_{min} = 0$ $g_{max} = 1$

$(0, 1)$ $(0, 0)$ $(1, 0)$ $(1, 1)$

0



1

Unanimously proposed

All correct processes eventually decide the unanimous proposal with high confidence value.

Graded Consensus

Asynchronous implementation with t<n/? Byzantine failures

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
  broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from [redacted] processes:
  broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n-t)/2$

upon ⟨ECHO, ·⟩ is received from [redacted] processes:
  if $\exists v'' \in$ Binary_Value, s.t. at least [redacted] ECHO messages contain value v'':
    trigger decide(v'', 1)
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```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

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Termination: Every correct process eventually triggers a proposal.

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PROOF.

Termination: Every correct process eventually triggers a proposal. Thus, every correct process eventually broadcasts a PROPOSAL message

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Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from $n - t$ processes:
    broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n - t)/2$

upon ⟨ECHO, ·⟩ is received from [redacted] processes:
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PROOF.

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Proof.

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upon ⟨ECHO, ·⟩ is received from $n - t$ processes:
  if $\exists v'' \in$ Binary_Value, s.t. at least [redacted] ECHO messages contain value v'':
    trigger decide(v'', 1)
  else:
    trigger decide(v^*, 0), where v^* denotes the value with the highest frequency among the ECHO messages. ▷ $\#(v^*) > (n-t)/2$

LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Termination: Every correct process eventually triggers a proposal. Thus, every correct process eventually broadcasts a PROPOSAL message and receives $(n - t)$ PROPOSAL messages. Consequently, every correct process eventually broadcasts an ECHO message and receives $(n - t)$ ECHO messages before triggering a decision.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from $n - t$ processes:
    broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n-t)/2$

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Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Unanimity Property:

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon $\boxed{\text{propose}(v_i \in \text{Binary\_Value})}$
  broadcast $\langle$PROPOSAL, $v_i\rangle$

upon $\langle$PROPOSAL, $\cdot\rangle$ is received from $n - t$ processes:
  broadcast $\langle$ECHO, $v'\rangle$, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n - t)/2$

upon $\langle$ECHO, $\cdot\rangle$ is received from $n - t$ processes:
  if $\exists v'' \in$ Binary_Value, s.t. at least [redacted] ECHO messages contain value v'':
    trigger decide$(v'', 1)$
  else:
    trigger decide$(v^*, 0)$, where v^* denotes the value with the highest frequency among the ECHO messages. ▷ $\#(v^*) > (n - t)/2$

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Unanimity Property: Suppose all correct processes propose the same value v.

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upon propose($v_i \in$ Binary_Value):
  $\boxed{\textbf{broadcast}\ \langle \text{PROPOSAL}, v_i \rangle}$

upon ⟨PROPOSAL, ·⟩ is received from $n - t$ processes:
  broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n - t)/2$

upon ⟨ECHO, ·⟩ is received from $n - t$ processes:
  if $\exists v'' \in$ Binary_Value, s.t. at least [redacted] ECHO messages contain value v'':
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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v.

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upon propose($v_i \in$ Binary_Value):
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Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages,

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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages, including at least $\boxed{(n - 2t)}$ with value $\boxed{v}$ and at most $\boxed{t}$ with value $\boxed{1 - v}$.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩
upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
    broadcast ⟨ECHO, v′⟩, where v′ denotes the value with the highest frequency among the PROPOSAL messages.    ▷ #(v′) > (n − t)/2
upon ⟨ECHO, ·⟩ is received from n − t processes:
    if ∃v″ ∈ Binary_Value, s.t. at least [redacted] ECHO messages contain value v″:
        trigger decide(v″, 1)
    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.    ▷ #(v*) > (n − t)/2
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Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n-t)$ PROPOSAL messages, including at least $\boxed{(n-2t)}$ with value $\boxed{v}$ and at most $\boxed{t}$ with value $\boxed{1-v}$. Given that $n > 3t$,

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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upon propose(v_i ∈ Binary_Value):
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upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
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upon ⟨ECHO, ·⟩ is received from n − t processes:
    if ∃v″ ∈ Binary_Value, s.t. at least [redacted] ECHO messages contain value v″:
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    else:
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```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages, including at least $(n - 2t)$ with value v and at most t with value $1 - v$. Given that $n > 3t$, we have $(n - 2t) > t$, ensuring

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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upon propose(v_i ∈ Binary_Value):
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upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
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upon ⟨ECHO, ·⟩ is received from n − t processes:
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Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages, including at least $(n - 2t)$ with value v and at most t with value $1 - v$. Given that $n > 3t$, we have $(n - 2t) > t$, ensuring that every correct process broadcasts an ECHO message with value v.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
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upon ⟨ECHO, ·⟩ is received from n − t processes:
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        trigger decide(v'', 1)
    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.    ▷ #(v*) > (n − t)/2
```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages, including at least $(n - 2t)$ with value v and at most t with value $1 - v$. Given that $n > 3t$, we have $(n - 2t) > t$, ensuring that every correct process broadcasts an ECHO message with value v. Thus, each correct process receives $(n - t)$ ECHO messages,

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
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upon $\langle$PROPOSAL, $\cdot\rangle$ is received from $n - t$ processes:
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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n-t)$ PROPOSAL messages, including at least $(n-2t)$ with value v and at most t with value $1-v$. Given that $n > 3t$, we have $(n-2t) > t$, ensuring that every correct process broadcasts an ECHO message with value v. Thus, each correct process receives $(n-t)$ ECHO messages, including at least [redacted] with value v,

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n - t)$ PROPOSAL messages, including at least $(n - 2t)$ with value v and at most t with value $1 - v$. Given that $n > 3t$, we have $(n - 2t) > t$, ensuring that every correct process broadcasts an ECHO message with value v. Thus, each correct process receives $(n - t)$ ECHO messages, including at least $(n - 2t)$ with value v,

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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Proof.

Unanimity Property: Suppose all correct processes propose the same value v. Then, each correct process broadcasts a PROPOSAL message with value v. Consequently, each process eventually receives $(n-t)$ PROPOSAL messages, including at least $(n-2t)$ with value v and at most t with value $1-v$. Given that $n > 3t$, we have $(n-2t) > t$, ensuring that every correct process broadcasts an ECHO message with value v. Thus, each correct process receives $(n-t)$ ECHO messages, including at least $(n-2t)$ with value v, and consequently decides on $(v, 1)$.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
  broadcast ⟨PROPOSAL, v_i⟩

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Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Consistency:

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

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    else:
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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$

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upon $\langle$ECHO, $\cdot\rangle$ is received from $n - t$ processes:
    if $\exists v'' \in$ Binary_Value, s.t. at least $n - 2t$ ECHO messages contain value v'':
        $\boxed{\textbf{trigger}\ \text{decide}(v'', 1)}$
    else:
        trigger decide($v^*, 0$), where v^* denotes the value with the highest frequency among the ECHO messages. ▷ $\#(v^*) > (n-t)/2$

LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate).

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upon propose(v_i ∈ Binary_Value):
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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes.

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PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j,

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upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
    broadcast ⟨ECHO, v′⟩, where v′ denotes the value with the highest frequency among the PROPOSAL messages.   ▷ #(v′) > (n − t)/2

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Proof.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$.

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upon propose($v_i \in$ Binary_Value):
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Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

```
upon propose(v_i ∈ Binary_Value):
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Proof.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

Process p_j's decision was based on receiving ECHO messages from a set Q_j with $|Q_j| = n - t$ processes.

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
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upon ⟨PROPOSAL, $\cdot$⟩ is received from $n - t$ processes:
    broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n - t)/2$

upon ⟨ECHO, $\cdot$⟩ is received from $n - t$ processes:
    if $\exists v'' \in$ Binary_Value, s.t. at least $n - 2t$ ECHO messages contain value v'':
        trigger decide(v'', 1)
    else:
        trigger decide(v^*, 0), where v^* denotes the value with the highest frequency among the ECHO messages. ▷ $\#(v^*) > (n - t)/2$

LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

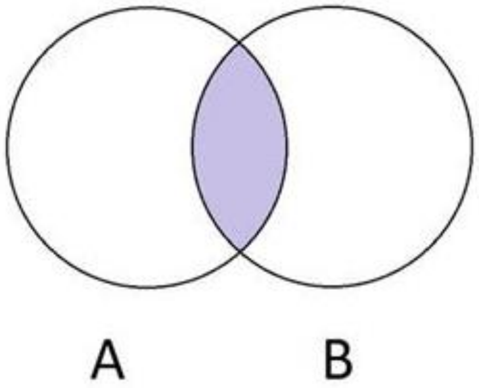
Process p_j's decision was based on receiving ECHO messages from a set Q_j with $|Q_j| = n - t$ processes. The overlap between Q_i and Q_j is $|Q_i \cap Q_j| =$

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

upon propose($v_i \in$ Binary_Value):
  broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, $\cdot$⟩ is received from $n - t$ processes:
  broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages.

upon ⟨ECHO, $\cdot$⟩ is received from $n - t$ processes:
  if $\exists v'' \in$ Binary_Value, s.t. at least $n - 2t$ ECHO messages contain value v'':
    trigger decide(v'', 1)
  else:
    trigger decide(v^*, 0), where v^* denotes the value with the highest frequency among the ECHO messages.



LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

Process p_j's decision was based on receiving ECHO messages from a set Q_j with $|Q_j| = n - t$ processes. The overlap between Q_i and Q_j is $|Q_i \cap Q_j| = |Q_i| + |Q_j| - |Q_i \cup Q_j|$

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

```
upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
    broadcast ⟨ECHO, v′⟩, where v′ denotes the value with the highest frequency among the PROPOSAL messages.    ▷ #(v′) > (n − t)/2

upon ⟨ECHO, ·⟩ is received from n − t processes:
    if ∃v″ ∈ Binary_Value, s.t. at least n − 2t ECHO messages contain value v″:
        trigger decide(v″, 1)
    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.    ▷ #(v*) > (n − t)/2
```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

Proof.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

Process p_j's decision was based on receiving ECHO messages from a set Q_j with $|Q_j| = n - t$ processes. The overlap between Q_i and Q_j is $|Q_i \cap Q_j| = |Q_i| + |Q_j| - |Q_i \cup Q_j| \geq (n - 2t) + (n - t) - n$

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upon propose($v_i \in$ Binary_Value):
 broadcast $\langle$PROPOSAL, $v_i\rangle$

upon $\langle$PROPOSAL, $\cdot\rangle$ is received from $n - t$ processes:
 broadcast $\langle$ECHO, $v'\rangle$, where v' denotes the value with the highest frequency among the PROPOSAL messages. ▷ $\#(v') > (n - t)/2$

upon $\langle$ECHO, $\cdot\rangle$ is received from $n - t$ processes:
 if $\exists v'' \in$ Binary_Value, s.t. at least $n - 2t$ ECHO messages contain value v'':
  trigger decide(v'', 1)
 else:
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LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

Process p_j's decision was based on receiving ECHO messages from a set Q_j with $|Q_j| = n - t$ processes. The overlap between Q_i and Q_j is $|Q_i \cap Q_j| = |Q_i| + |Q_j| - |Q_i \cup Q_j| \geq (n - 2t) + (n - t) - n = n - 3t$,

Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/X$ Pseudocode (for process p_i)

```
upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
    broadcast ⟨ECHO, v′⟩, where v′ denotes the value with the highest frequency among the PROPOSAL messages.    ▷ #(v′) > (n − t)/2

upon ⟨ECHO, ·⟩ is received from n − t processes:
    if ∃v′′ ∈ Binary_Value, s.t. at least n − 2t ECHO messages contain value v′′:
        trigger decide(v′′, 1)
    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.    ▷ #(v*) > (n − t)/2
```

LEMMA 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

PROOF.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

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upon propose($v_i \in$ Binary_Value):
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Proof.

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Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/$X Pseudocode (for process p_i)

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upon propose(v_i ∈ Binary_Value):
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upon propose(v_i ∈ Binary_Value):
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```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and n/X-resiliency.*

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Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/7$: Pseudocode (for process p_i)

```
upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
    broadcast ⟨ECHO, v'⟩, where v' denotes the value with the highest frequency among the PROPOSAL messages.   ▷ #(v') > (n − t)/2

upon ⟨ECHO, ·⟩ is received from n − t processes:
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        trigger decide(v'', 1)
    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.   ▷ #(v*) > (n − t)/2
```

Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and $n/7$-resiliency.*

Proof.

Consistency: Assume a correct process p_i decides $(w, 1)$ (otherwise, consistency is immediate). Process p_i must have received ECHO messages with value w from a set Q_i of $|Q_i| = n - 2t$ distinct processes. Let another correct process, p_j, decide on some value $(w', \cdot)$. We aim to show that $w' = w$.

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Algorithm 1 Asynchronous Binary Graded Consensus with refinement $R = 2$, and $t < n/7$: Pseudocode (for process p_i)

```
upon propose(v_i ∈ Binary_Value):
    broadcast ⟨PROPOSAL, v_i⟩

upon ⟨PROPOSAL, ·⟩ is received from n − t processes:
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    else:
        trigger decide(v*, 0), where v* denotes the value with the highest frequency among the ECHO messages.   ▷ #(v*) > (n − t)/2
```

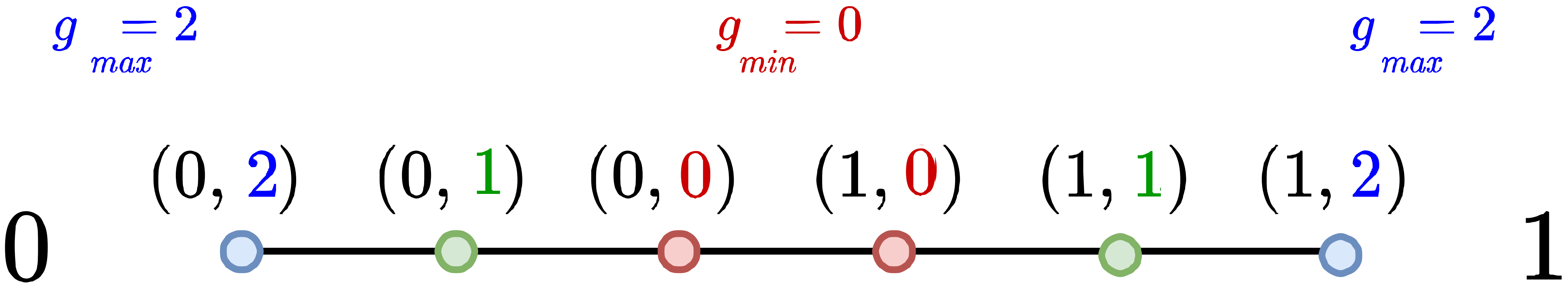
Lemma 1.1. *Algorithm 1 implements graded consensus with the refinement parameter $R = 2$ and $n/7$-resiliency.*

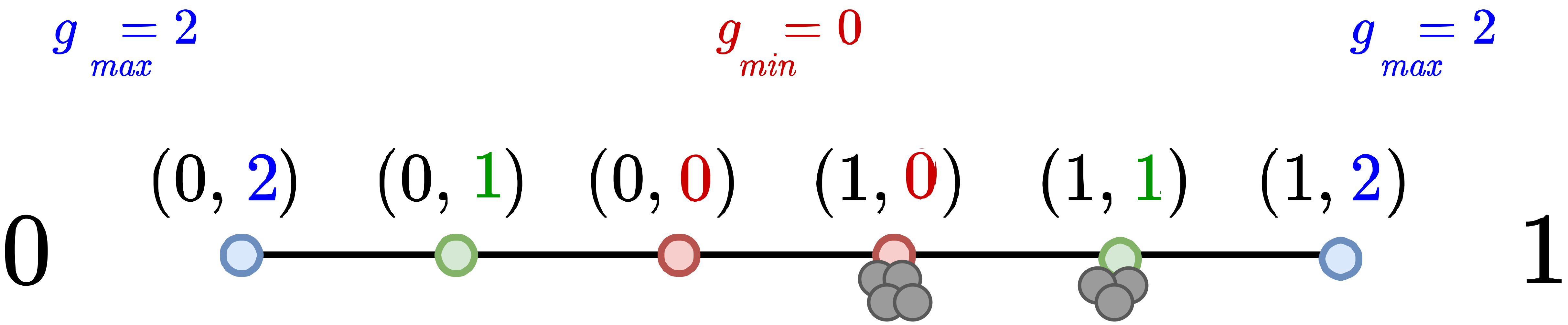
$$\boxed{Alg_1} \sqsubseteq \boxed{GC^{R=2}}$$

One more refinement

Properties of Graded Consensus

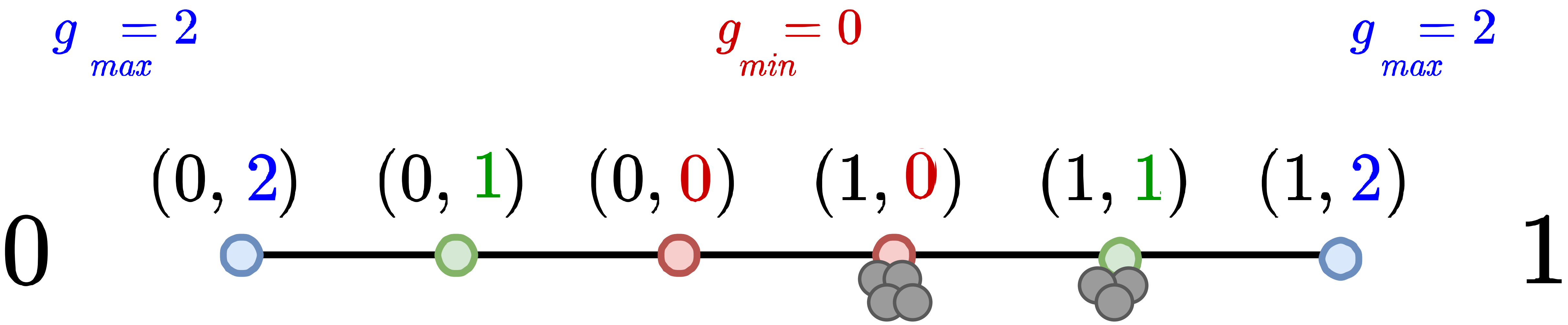
- **Termination:** All correct processes decide.
- **Integrity:** No process decides more than once.
- **Consistency (~~Agreement~~):** Assume two correct processes decide (v, g) and (v', g'). We have (a) |g-g'|≤1, and (b) if v≠v', then g=g'=0.
- **Unanimity:** If only v is proposed, then only (v, g_{max}) can be decided.


$g_{max} = 2$
$g_{min} = 0$
$g_{max} = 2$
$(0, 2)$
$(0, 1)$
$(0, 0)$
$(1, 0)$
$(1, 1)$
$(1, 2)$
0
1



(1) If two correct processes decide (v,g) and (v',g'), then **either** (i) g=g'=0, **or** (ii) |g-g'|≤1, and v=v'.

(2) If two correct processes decide (v,g) and (v',g'), then (a) |g-g'|≤1, **and** (b) (v≠v' ⟹ g=g'=0).



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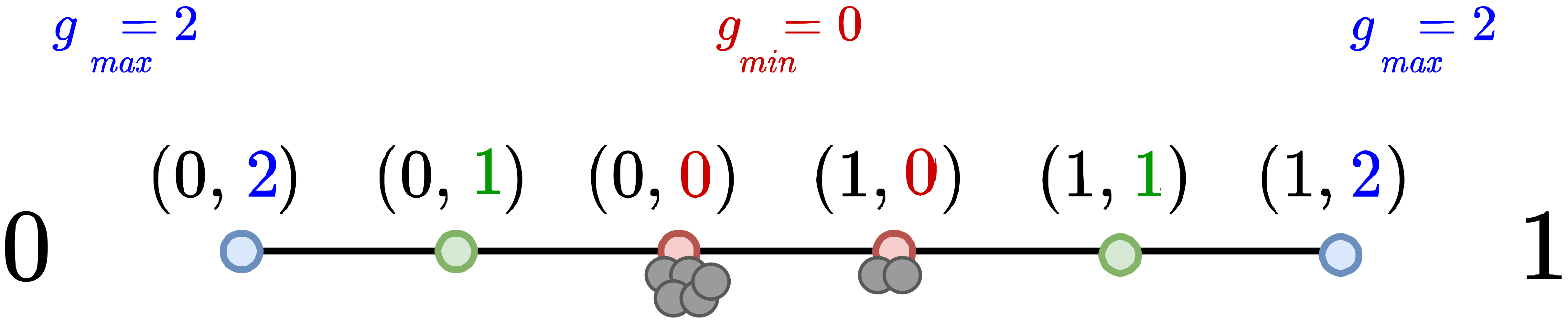
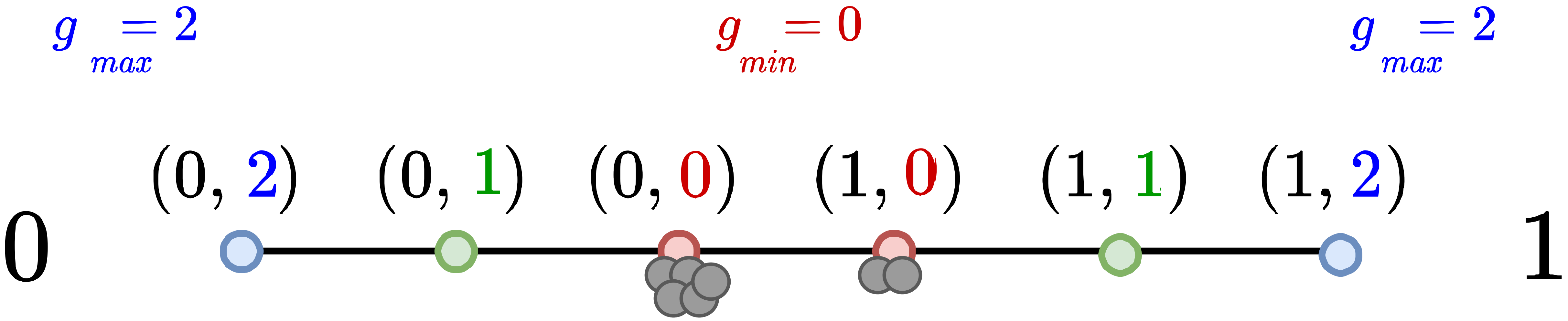


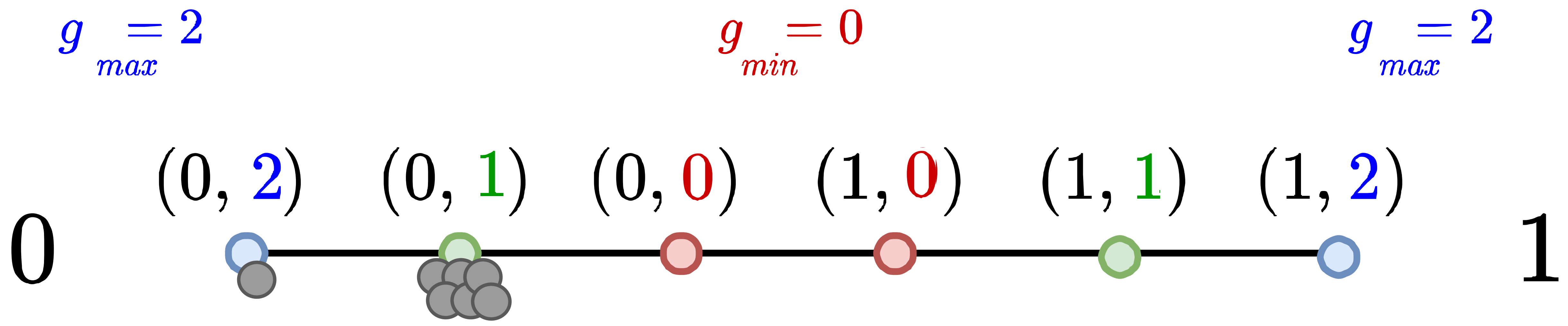
(1) If two correct processes decide (v,g) and (v’,g’), then **either** (i) g=g’=0, **or** (ii) |g-g’|≤1, and v=v’.

(2) If two correct processes decide (v,g) and (v’,g’), then (a) |g-g’|≤1, **and** (b) (v≠v’ ⟹ g=g’=0).



(1) If two correct processes decide (v,g) and (v',g'), then **either** (i) $\boxed{g=g'=0}$, **or** (ii) $|g-g'| \leq 1$, and v=v'.

(2) If two correct processes decide (v,g) and (v',g'), then (a) $|g-g'| \leq 1$, **and** (b) ($v \neq v' \Rightarrow g=g'=0$).



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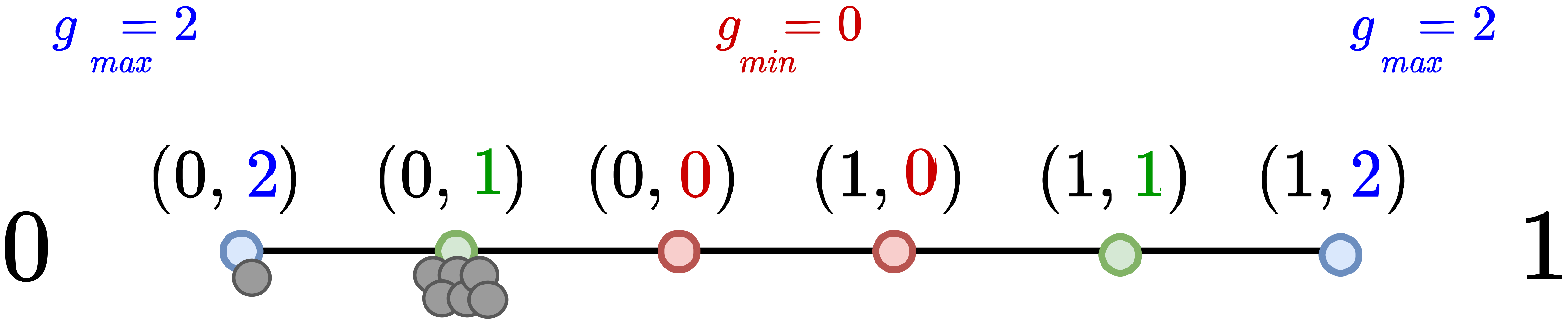

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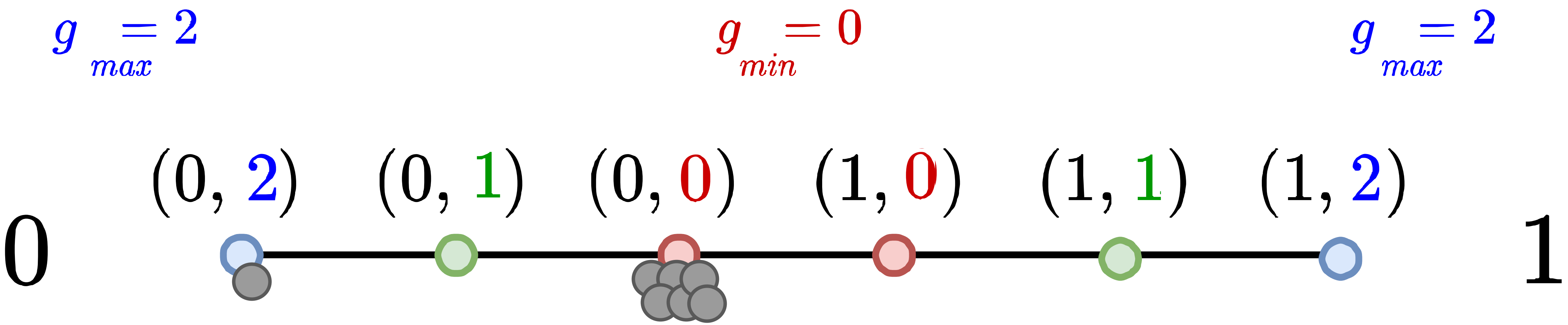

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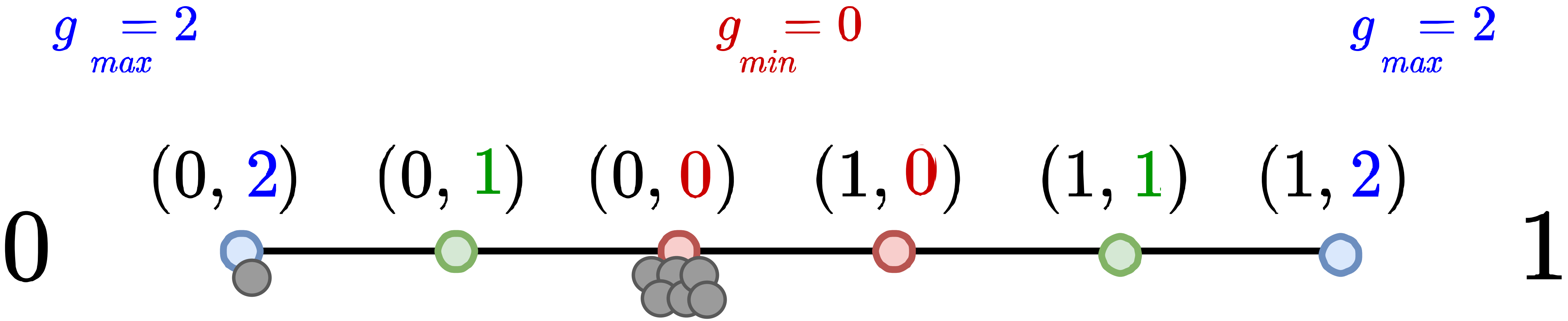

(1) If two correct processes decide (v,g) and (v',g'), then **either** (i) g=g'=0, **or** (ii) |g-g'|≤1, and v=v'.

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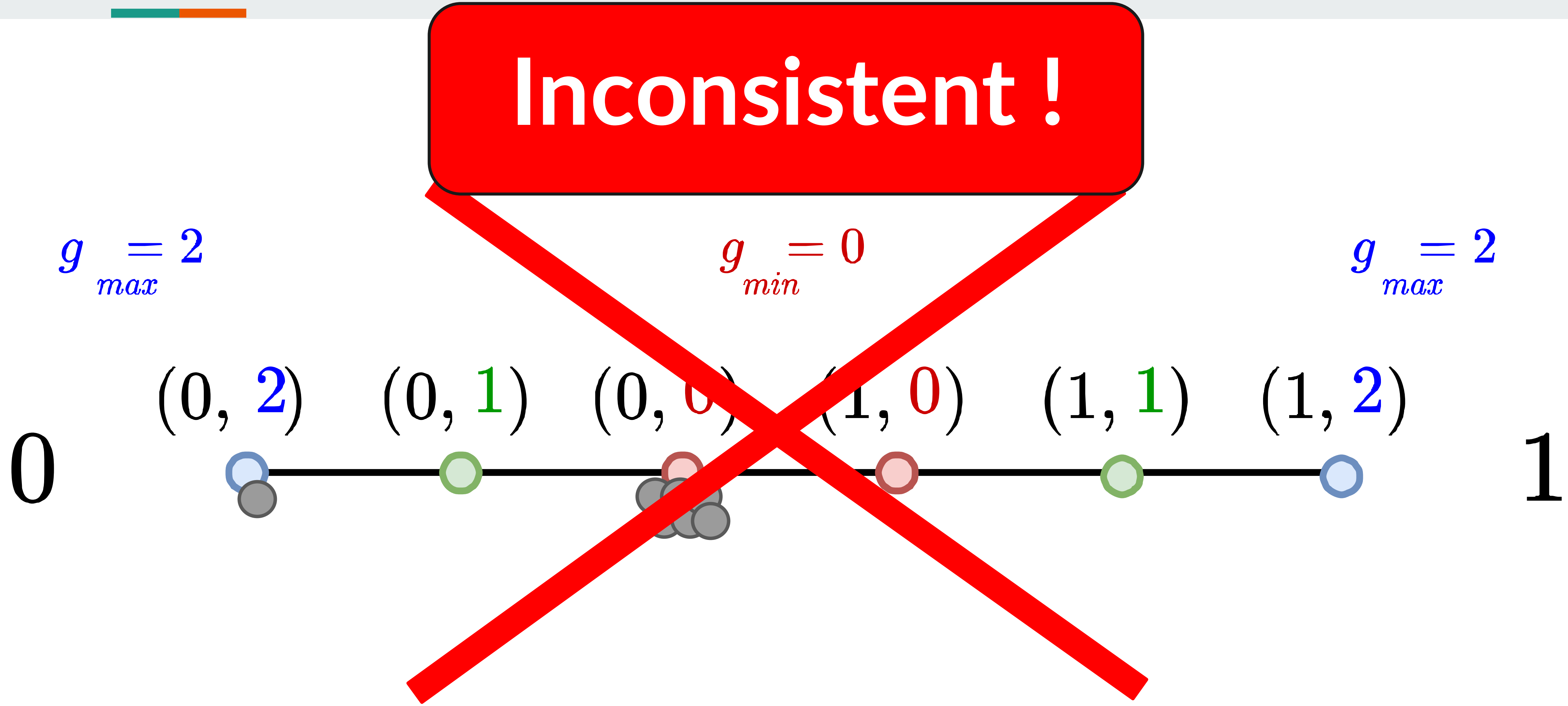

(1) If two correct processes decide (v,g) and (v',g'), then **either** (i) g=g'=0, **or** (ii) |g-g'|≤1, and v=v'.

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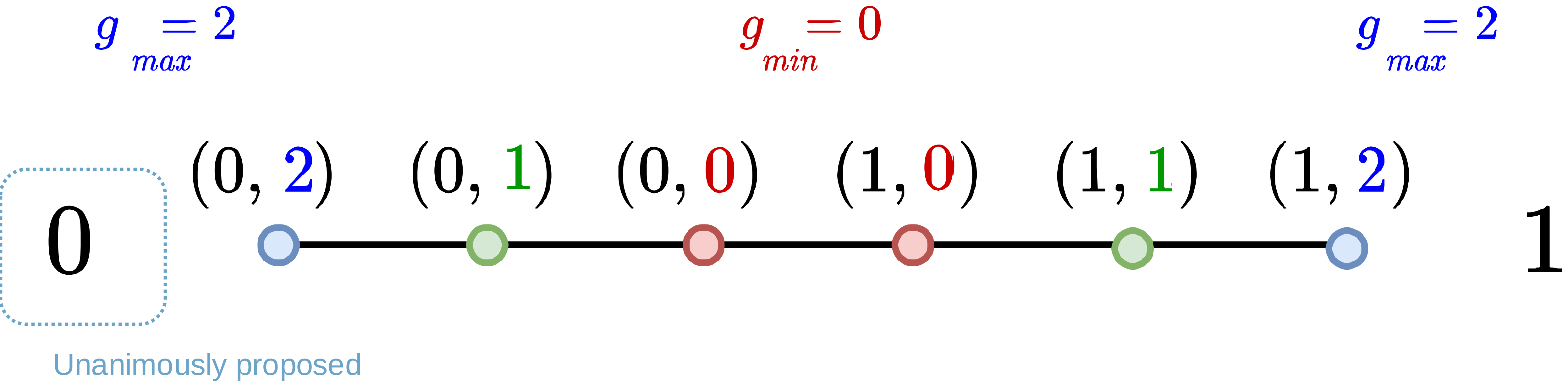
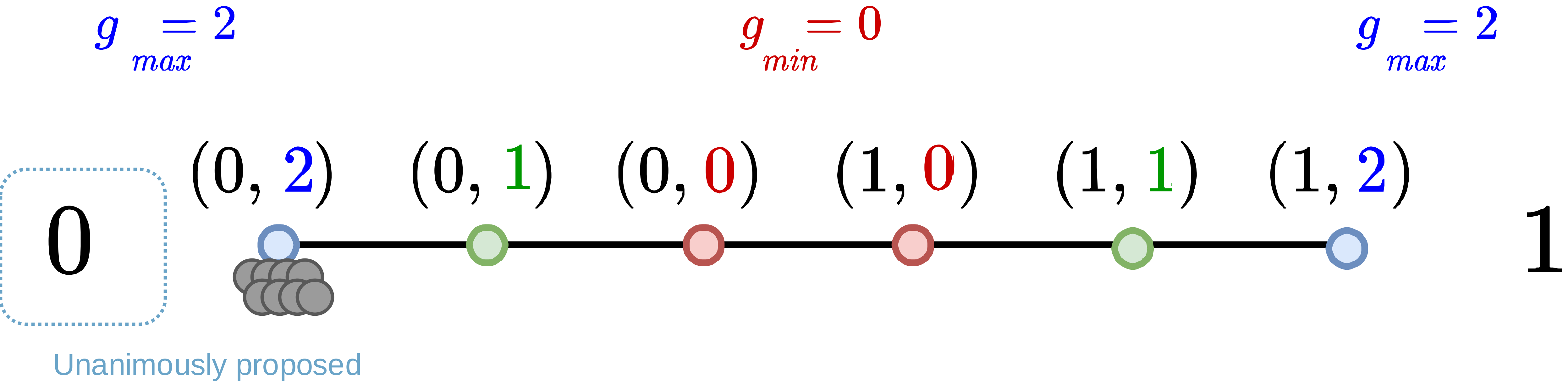


All correct processes eventually decide the unanimous proposal with high confidence value.



All correct processes eventually decide the unanimous proposal with high confidence value.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
    Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
    Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
    invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
    upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
        [redacted] ▷ Update the grade (confidence)
        invoke $\mathcal{GC}_2$.propose([redacted]) ▷ Propose input value to the 2nd graded consensus instance
    upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
        [redacted] ▷ Update the grade (confidence)
        trigger decide([redacted]) ▷ Decide the final value and grade

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    [redacted] ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    [redacted] ▷ Update the grade (confidence)
    trigger decide(v_i^2, [redacted]) ▷ Decide the final value and grade

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    [redacted] ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    [redacted] ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

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Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

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Local Variables:
  Integer $g_i \leftarrow 0$ ▹ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▹ Propose input value to the 1st graded consensus instance

  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▹ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▹ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▹ Propose input value to the 2nd graded consensus instance

  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▹ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▹ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▹ Decide the final value and grade

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
Binary Graded Consensus, **instances** $\mathcal{GC}_1$, $\mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$
Local Variables:
Integer $g_i \leftarrow 0$ ▷ Grade
upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

Proof.

- Termination follows directly from the termination of $\mathcal{GC}_1$ and $\mathcal{GC}_2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$
Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade
upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

PROOF.

- Termination follows directly from the termination of $\mathcal{GC}_1$ and $\mathcal{GC}_2$.
- Unanimity property follows directly from the unanimity property of $\mathcal{GC}_1$ and $\mathcal{GC}_2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

```
Uses:
    Binary Graded Consensus, instances GC_1, GC_2   ▷ 2 instances of the Binary Graded Consensus protocol with Refinement R' = 2
Local Variables:
    Integer g_i ← 0                                  ▷ Grade
upon propose(v_i ∈ Value):
    invoke GC_1.propose(v_i)                         ▷ Propose input value to the 1st graded consensus instance
    upon GC_1.decide(v_i^1, g_i^1):                  ▷ Received decision from the 1st graded consensus instance
        g_i ← g_i + g_i^1                            ▷ Update the grade (confidence)
        invoke GC_2.propose(v_i^1)                   ▷ Propose input value to the 2nd graded consensus instance
    upon GC_2.decide(v_i^2, g_i^2):                  ▷ Received decision from the 2nd graded consensus instance
        g_i ← g_i + g_i^2                            ▷ Update the grade (confidence)
        trigger decide(v_i^2, g_i)                   ▷ Decide the final value and grade
```

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

  upon $\boxed{\mathcal{GC}_1}$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

1  $(v_i^1 = w, 1)$!

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

$\boxed{\text{Case 1: } j = 1.}$

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
 Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
 Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
 invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

$(v_i^1 = w, 1)$!

$(v_j^1 = w, *)$!

 upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
  $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
  invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

 upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
  $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
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LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 1: $j = 1$. By consistency, each correct process outputs $(w, \cdot)$ from $\mathcal{GC}_1$ for some value w, and thus proposes w to $\mathcal{GC}_2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
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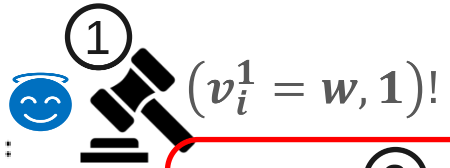
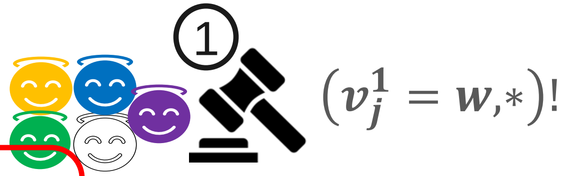
Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance





  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 1: $j = 1$. By consistency, each correct process outputs $(w, \cdot)$ from $\mathcal{GC}_1$ for some value w, and thus proposes w to $\mathcal{GC}_2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$
Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade
upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) 1 $(v_i^1 = w, 1)$! 1 $(v_j^1 = w, *)$! ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) 2 w! ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): 2 $(v_*^2 = w, 1)$! ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 1: $j = 1$. By consistency, each correct process outputs $(w, \cdot)$ from $\mathcal{GC}_1$ for some value w, and thus proposes w to $\mathcal{GC}_2$. Therefore, due to the unanimity property of graded consensus $\mathcal{GC}_2$, every correct process returns $(w, 1)$ from $\mathcal{GC}_2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▹ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$
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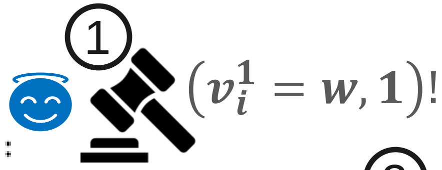
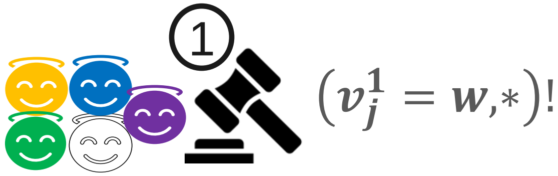
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▹ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▹ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▹ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▹ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▹ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▹ Decide the final value and grade





Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 1: $j = 1$. By consistency, each correct process outputs $(w, \cdot)$ from $\mathcal{GC}_1$ for some value w, and thus proposes w to $\mathcal{GC}_2$. Therefore, due to the unanimity property of graded consensus $\mathcal{GC}_2$, every correct process returns $(w, 1)$ from $\mathcal{GC}_2$. Hence, consistency follows directly from the consistency of $\mathcal{GC}_1$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$
Local Variables:
Integer $g_i \leftarrow 0$ ▷ Grade
upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\boxed{\mathcal{GC}_2}$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 2: $j = 2$.

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
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Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance
  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 2: $j = 2$. By construction, each correct process outputs $(\cdot, 0)$ from $\mathcal{GC}_1$.

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Uses:
Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
Integer $g_i \leftarrow 0$ ▷ Grade

upon propose$(v_i \in \text{Value})$:
invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

upon $\boxed{\mathcal{GC}_1}$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
$g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
$g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

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Local Variables:
Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

$(*, \mathbf{0})$!

upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
$g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
$g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

$(*,*)$!

Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 2: $j = 2$. By construction, each correct process outputs $(\cdot, 0)$ from $\mathcal{GC}_1$. Therefore, due to the consistency property of graded consensus $\mathcal{GC}_2$,

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
  Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
  Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
  invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
    invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance
  upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
    $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
    trigger decide(v_i^2, g_i) ▷ Decide the final value and grade

LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

Case 2: $j = 2$. By construction, each correct process outputs $(\cdot, 0)$ from $\mathcal{GC}_1$. Therefore, due to the consistency property of graded consensus $\mathcal{GC}_2$, if two correct processes p_i and p_j decide on (v_i, g_i) and (v_j, g_j), respectively, then

Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

Uses:
    Binary Graded Consensus, **instances** $\mathcal{GC}_1, \mathcal{GC}_2$ ▷ 2 instances of the Binary Graded Consensus protocol with Refinement $R' = 2$

Local Variables:
    Integer $g_i \leftarrow 0$ ▷ Grade

upon propose($v_i \in$ Value):
    invoke $\mathcal{GC}_1$.propose(v_i) ▷ Propose input value to the 1st graded consensus instance

    upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
        $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
        invoke $\mathcal{GC}_2$.propose(v_i^1) ▷ Propose input value to the 2nd graded consensus instance

    upon $\mathcal{GC}_2$.decide(v_i^2, g_i^2): ▷ Received decision from the 2nd graded consensus instance
        $g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
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$g_i \leftarrow g_i + g_i^2$ ▷ Update the grade (confidence)
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LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

- To prove consistency, let j be the first instance of graded consensus where some process outputs $(\cdot, 1)$ from $\mathcal{GC}_j$. If no such j exists, the result is immediate.

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Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

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Local Variables:
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upon propose($v_i \in$ Value):
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  upon $\mathcal{GC}_1$.decide(v_i^1, g_i^1): ▷ Received decision from the 1st graded consensus instance
    $g_i \leftarrow g_i + g_i^1$ ▷ Update the grade (confidence)
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LEMMA 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

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Lemma 1.2. *Algorithm 2 implements graded consensus with the refinement parameter $R = 3$.*

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Algorithm 2 Binary Graded Consensus (BGC) with refinement $R = 3$ on top of BGC with refinement $R = 2$

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$$\boxed{Alg_2} = \boxed{GC^{R=2}} \rhd \boxed{GC^{R=2}} \sqsubseteq \boxed{GC^{R=3}}$$

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        $g_i \leftarrow g_i + g_i^2$ ▹ Update the grade (confidence)
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$$\boxed{Alg_2} = \boxed{GC^{R=2}} \rhd \boxed{GC^{R=2}} \sqsubseteq \boxed{GC^{R=3}}$$

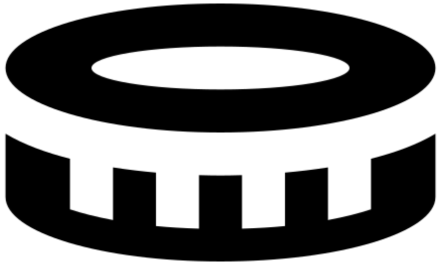
$$\boxed{Alg_2} \sqsubseteq \boxed{GC^{R=3}}$$

Common Coin

Interface of the common coin

Flip
Operation

Yield
Callback



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$\emptyset$

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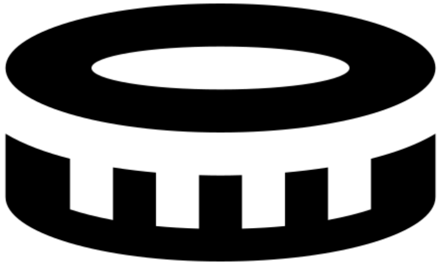

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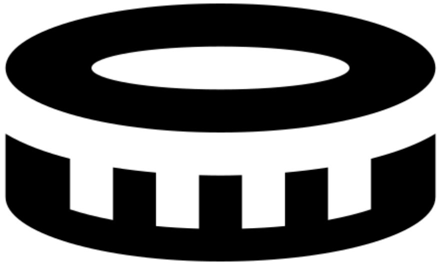

$v!$

Properties of the Common Coin

- **Termination:** All correct processes eventually yield a binary value.
- **Agreement:** All correct processes agree on 0 (or 1) with probability >0.
- **Unpredictability:** As soon no correct process has triggered ‘flip’, the adversary cannot predict the output with probability greater than 1/2.

A naive implementation of the Common Coin

Algorithm 3 Common Coin

1: **upon** flip():
2: $\quad b \xleftarrow{\$} \{0, 1\}$ ▷ Choose either 0 or 1 with probability 1/2
3: $\quad$ **trigger** yield(b)

A naive implementation of the Common Coin

Algorithm 3 Common Coin

1: **upon** flip():
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3: **trigger** yield(b)

$$\boxed{Alg_3} \sqsubseteq \boxed{CC}$$

Consensus = Stay safe + Try (and try again)

$$\boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \cdots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \cdots}$$

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

Uses:
 Binary "Extended" Graded Consensus, **instances** $\mathcal{EGC}_1, \mathcal{EGC}_2, \ldots$ ▹ ∞ instances of the Binary Graded Consensus with Refinement $R = 3$

Constants:
 Integer $g_{min} \leftarrow 0$
 Integer $g_{max} \leftarrow 2$

Local Variables:
 Binary_Value $est_i \leftarrow 0$ ▹ Estimate Value
 Integer $g_i \leftarrow g_{min}$ ▹ Grade (Confidence) in $\{0, 1, 2\}$
 Integer $attempt \leftarrow 0$
 Integer $halt \leftarrow \infty$
 Boolean $decided \leftarrow false$

upon propose($v_i \in$ Value):
 $est_i \leftarrow v_i$:
 while $halt \geq attempt$:
  //safety guard
  $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
  if [redacted]
   trigger decide(est_i) ▹ Decide
   $decided \leftarrow true$
   $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
  //try to converge
  $b_i \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
  if [redacted]
   $est_i \leftarrow b_i$
  $attempt \leftarrow attempt + 1$

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  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▹ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
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    $b_i \leftarrow \mathcal{CC}_{attempt}$.flip() ▷ Execute instance of common coin
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      $est_i \leftarrow b_i$
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Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
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            $est_i \leftarrow b_i$
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LEMMA 1.3. *If all correct processes begin attempt k with the same estimate value v, they will all decide on v by attempt k and halt by attempt $k + 1$.*

PROOF.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \boxed{\mathcal{EGC}_{attempt}.\text{propose}(est_i)}$ ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}.\text{flip}()$ ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

Lemma 1.3. *If all correct processes begin attempt k with the same estimate value v, they will all decide on v by attempt k and halt by attempt $k + 1$.*

Proof. By the unanimity property of $\mathcal{EGC}_k$,

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$:
    while $halt \geq attempt$:
        //safety guard
        $\boxed{(est_i, g_i)} \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
        if $\boxed{g_i == g_{max}} \wedge decided = false$:
            trigger decide(est_i) ▷ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

LEMMA 1.3. *If all correct processes begin attempt k with the same estimate value v, they will all decide on v by attempt k and halt by attempt $k + 1$.*

PROOF. By the unanimity property of $\mathcal{EGC}_k$, all correct processes return (v, g_{max}) from $\mathcal{EGC}_k$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $\boxed{g_i == g_{max} \wedge decided = false}$:
      $\boxed{\textbf{trigger}\ \text{decide}(est_i)}$ ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

LEMMA 1.3. *If all correct processes begin attempt k with the same estimate value v, they will all decide on v by attempt k and halt by attempt $k + 1$.*

PROOF. By the unanimity property of $\mathcal{EGC}_k$, all correct processes return (v, g_{max}) from $\mathcal{EGC}_k$. The rest follows directly from the protocol. □

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
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    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      $\boxed{\textbf{trigger}\ \text{decide}(est_i)}$ ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k + 1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
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    $\boxed{(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}.\text{propose}(est_i)}$ ▷ Execute instance of extended graded consensus
    if $\boxed{g_i == g_{max}} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}.\text{flip}()$ ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k + 1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $\boxed{(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}.\text{propose}(est_i)}$ ▷ Execute instance of extended graded consensus
    if $\boxed{g_i == g_{max}} \wedge decided = false$:
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      $decided \leftarrow true$
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    //try to converge
    $b_i \leftarrow CC_{attempt}.\text{flip}()$ ▷ Execute instance of common coin
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    $attempt \leftarrow attempt + 1$

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k + 1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1, 2\})$ from $\mathcal{EGC}_k$

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$:
    while $halt \geq attempt$:
        //safety guard
        $\boxed{(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}.\text{propose}(est_i)}$ ▹ Execute instance of extended graded consensus
        if $\boxed{g_i == g_{max}} \wedge decided = false$:
            trigger decide(est_i) ▹ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        $b_i \leftarrow CC_{attempt}.\text{flip}()$ ▹ Execute instance of common coin
        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

LEMMA 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k+1$.*

PROOF. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1,2\})$ from $\mathcal{EGC}_k$, updating its estimate est_j to v.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)          ▷ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                           ▷ Decide
            decided ← true
            halt ← attempt + 1                              ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                             ▷ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k+1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1, 2\})$ from $\mathcal{EGC}_k$, updating its estimate est_j to v. Thus, $g_j > g_{min}$,

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    $\boxed{\textbf{if } g_i == g_{min}}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

LEMMA 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k+1$.*

PROOF. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1, 2\})$ from $\mathcal{EGC}_k$, updating its estimate est_j to v. Thus, $g_j > g_{min}$, so all correct processes ignore the output of CC_k and retain $est_j = v$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$:
    while $halt \geq attempt$:
        //safety guard
        $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
        if $g_i == g_{max} \land decided = false$:
            trigger decide(est_i) ▷ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k+1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1, 2\})$ from $\mathcal{EGC}_k$, updating its estimate est_j to v. Thus, $g_j > g_{min}$, so all correct processes ignore the output of CC_k and retain $est_j = v$. Consequently, all correct processes begin attempt $k+1$ with estimate value v. □

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \land decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

Lemma 1.4. *If a correct process decides on v in attempt k, then all correct processes will decide on v by attempt $k + 1$.*

Proof. Let p_i be the first correct process to decide, and assume it decides on v at attempt k. This implies that p_i returned (v, g_{max}) from $\mathcal{EGC}_k$. By the consistency property of $\mathcal{EGC}_k$, every correct process p_j returns $(v, g_j \in \{1, 2\})$ from $\mathcal{EGC}_k$, updating its estimate est_j to v. Thus, $g_j > g_{min}$, so all correct processes ignore the output of CC_k and retain $est_j = v$. Consequently, all correct processes begin attempt $k + 1$ with estimate value v. Lemma 1.3 then completes the proof. □

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)        ▷ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                       ▷ Decide
            decided ← true
            halt ← attempt + 1                          ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                         ▷ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

PROOF.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$:
    while $halt \geq attempt$:
        //safety guard
        $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$)   ▷ Execute instance of extended graded consensus
        if $g_i == g_{max} \wedge decided = false$:
            trigger decide(est_i)   ▷ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        $b_i \leftarrow CC_{attempt}$.flip()   ▷ Execute instance of common coin
        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

Proof. **Validity** follows from

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

PROOF. **Validity** follows from Lemma 1.3,

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$:
    while $halt \geq attempt$:
        //safety guard
        $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
        if $g_i == g_{max} \wedge decided = false$:
            trigger decide(est_i) ▷ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

PROOF. **Validity** follows from Lemma 1.3, and **Agreement** follows from

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)      ▹ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                      ▹ Decide
            decided ← true
            halt ← attempt + 1                         ▹ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                        ▹ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

PROOF. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)          ▹ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                          ▹ Decide
            decided ← true
            halt ← attempt + 1                             ▹ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                            ▹ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

Proof. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4. We now prove **Termination**.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▹ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

PROOF. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4. We now prove **Termination**. Let p_i be the first correct process calling CC_k for some attempt k.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)      ▷ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                      ▷ Decide
            decided ← true
            halt ← attempt + 1                         ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                        ▷ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

(b, g_i)!

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

Proof. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4. We now prove **Termination**. Let p_i be the first correct process calling CC_k for some attempt k. Let (b, g_i) the pair returned by p_i from $\mathcal{EGC}_k$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$;
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$)   ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i)   ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $\boxed{b_i} \leftarrow CC_{attempt}$.flip()   ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

$(\boxed{b}, g_i)$!

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

PROOF. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4. We now prove **Termination**. Let p_i be the first correct process calling CC_k for some attempt k. Let (b, g_i) the pair returned by p_i from $\mathcal{EGC}_k$. With non-zero probability ρ, all correct processes return b from CC_k.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
 $est_i \leftarrow v_i$;
 while $halt \geq attempt$:
  //safety guard
  $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($\boxed{est_i}$) ▷ Execute instance of extended graded consensus
  $(\boxed{b}, \boxed{g_i})$!
  if $g_i == g_{max} \wedge decided = false$:
   trigger decide(est_i) ▷ Decide
   $decided \leftarrow true$
   $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
  //try to converge
  $\boxed{b_i} \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
  if $g_i == g_{min}$:
   $est_i \leftarrow b_i$
  $attempt \leftarrow attempt + 1$

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

Proof. **Validity** follows from Lemma 1.3, and **Agreement** follows from Lemma 1.4. We now prove **Termination**. Let p_i be the first correct process calling CC_k for some attempt k. Let (b, g_i) the pair returned by p_i from $\mathcal{EGC}_k$. With non-zero probability ρ, all correct processes return b from CC_k. We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)        ▷ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                          ▷ Decide
            decided ← true
            halt ← attempt + 1                             ▷ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                            ▷ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

(b, g_i)!

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$;
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    $(b, g_i)!$
    $(b, \boxed{g_j > g_{min}})!$
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow \mathcal{CC}_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\boxed{\exists p_j \in \text{Correct}, g_j > g_{min}}$:

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \boxed{\mathcal{EGC}_{attempt}}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow \mathcal{CC}_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

(b, g_i)!

$(b, \boxed{g_j > g_{min}})$!

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\boxed{\exists p_j \in \text{Correct}, g_j > g_{min}}$: By $\mathcal{EGC}_k$'s consistency,

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$;
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

$(b, g_i)!$

$(b, g_j > g_{min})!$

$(b, *)!$

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\exists p_j \in \text{Correct}, g_j > g_{min}$: By $\mathcal{EGC}_k$'s consistency, (a) all processes return $(b, \cdot)$ from $\mathcal{EGC}_k$,

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(\boxed{est_i, g_i}) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▹ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $\boxed{b_i} \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
    if $g_i == g_{min}$:
      $\boxed{est_i \leftarrow b_i}$
    $attempt \leftarrow attempt + 1$

$(\boxed{b,} g_i)!$

$(b, g_j > g_{min})!$

$(b, *)!$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\boxed{\exists p_j \in \text{Correct}, g_j > g_{min}}$: By $\mathcal{EGC}_k$'s consistency, (a) all processes return $(b, \cdot)$ from $\mathcal{EGC}_k$, so they will all have the same estimate value at attempt $k+1$, either by adopting CC_k's output or by retaining the value from $\mathcal{EGC}_k$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$;
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▹ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

(b, g_i)!

$(b, g_j > g_{min})$!

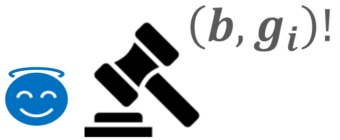
$(b, *)$!

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\exists p_j \in \text{Correct}, g_j > g_{min}$: By $\mathcal{EGC}_k$'s consistency, (a) all processes return $(b, \cdot)$ from $\mathcal{EGC}_k$, so they will all have the same estimate value at attempt $k+1$, either by adopting CC_k's output or by retaining the value from $\mathcal{EGC}_k$.

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$;
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▷ Execute instance of extended graded consensus


    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▷ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▷ Halt after the next attempt after having helped the remaining processes to decide


    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▷ Execute instance of common coin

    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

THEOREM 1.5. *Algorithm 4 implements binary Byzantine consensus with probability 1 and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

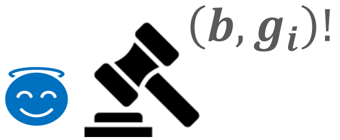
- **Case** $\exists p_j \in$ Correct, $g_j > g_{min}$: By $\mathcal{EGC}_k$'s consistency, (a) all processes return $(b, \cdot)$ from $\mathcal{EGC}_k$, so they will all have the same estimate value at attempt $k+1$, either by adopting CC_k's output or by retaining the value from $\mathcal{EGC}_k$.

- **Case** $\boxed{\forall p_j \in \text{Correct}, g_j = g_{min}}$:

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

---

upon propose($v_i \in$ Value):
    $est_i \leftarrow v_i$;
    while $halt \geq attempt$:
        //safety guard
        $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($\boxed{est_i}$) ▹ Execute instance of extended graded consensus



        if $g_i == g_{max} \wedge decided = false$:
            trigger decide(est_i) ▹ Decide
            $decided \leftarrow true$
            $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide

        //try to converge
        $\boxed{b_i} \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin

        if $g_i == g_{min}$:
            $est_i \leftarrow b_i$
        $attempt \leftarrow attempt + 1$

Theorem 1.5. *Algorithm 4 implements binary Byzantine consensus with probability* 1 *and has the same resiliency as the underlying graded consensus object.*

We consider two cases depending on the grades obtained by correct processes from $\mathcal{EGC}_k$. In both cases, all correct processes start the next attempt with the same estimate value, allowing us to apply Lemma 1.3.

- **Case** $\exists p_j \in \text{Correct}, g_j > g_{min}$: By $\mathcal{EGC}_k$'s consistency, (a) all processes return $(b, \cdot)$ from $\mathcal{EGC}_k$, so they will all have the same estimate value at attempt $k+1$, either by adopting CC_k's output or by retaining the value from $\mathcal{EGC}_k$.

- **Case** $\boxed{\forall p_j \in \text{Correct}, g_j = g_{min}}$: All processes adopt the value provided by the common coin, which is identical across processes. □

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

```
upon propose(v_i ∈ Value):
    est_i ← v_i:
    while halt ≥ attempt:
        //safety guard
        (est_i, g_i) ← EGC_attempt.propose(est_i)      ▹ Execute instance of extended graded consensus
        if g_i == g_max ∧ decided = false:
            trigger decide(est_i)                       ▹ Decide
            decided ← true
            halt ← attempt + 1                          ▹ Halt after the next attempt after having helped the remaining processes to decide
        //try to converge
        b_i ← CC_attempt.flip()                         ▹ Execute instance of common coin
        if g_i == g_min:
            est_i ← b_i
        attempt ← attempt + 1
```

$$\boxed{Alg_4} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \dots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \dots} \sqsubseteq \boxed{BA}$$

Algorithm 4 Byzantine Agreement Protocol with Extended Graded Consensus

upon propose($v_i \in$ Value):
  $est_i \leftarrow v_i$:
  while $halt \geq attempt$:
    //safety guard
    $(est_i, g_i) \leftarrow \mathcal{EGC}_{attempt}$.propose($est_i$) ▹ Execute instance of extended graded consensus
    if $g_i == g_{max} \wedge decided = false$:
      trigger decide(est_i) ▹ Decide
      $decided \leftarrow true$
      $halt \leftarrow attempt + 1$ ▹ Halt after the next attempt after having helped the remaining processes to decide
    //try to converge
    $b_i \leftarrow CC_{attempt}$.flip() ▹ Execute instance of common coin
    if $g_i == g_{min}$:
      $est_i \leftarrow b_i$
    $attempt \leftarrow attempt + 1$

$$\boxed{Alg_4} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \dots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \dots} \sqsubseteq \boxed{BA}$$

$$\boxed{GC^{R=2}} \rhd \boxed{GC^{R=2}} \sqsubseteq \boxed{GC^{R=3}}$$

Positive Result 1: *There exists* a **randomized asynchronous** protocol that solves consensus, while tolerating arbitrary (**Byzantine**) failures

A general perspective

$$\boxed{Alg_4} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \cdots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{CC} \longrightarrow \cdots} \sqsubseteq \boxed{BA}$$

$$\boxed{Alg'} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \cdots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \cdots} \sqsubseteq \boxed{BA}$$

Safety Guards

$$Alg' = \boxed{GC^{R=3} \longrightarrow Try \longrightarrow \dots \longrightarrow GC^{R=3} \longrightarrow Try \longrightarrow \dots} \sqsubseteq BA$$

Safety Guards

$$Alg' = \boxed{GC^{R=3} \longrightarrow Try \longrightarrow \dots \longrightarrow GC^{R=3} \longrightarrow Try \longrightarrow \dots} \sqsubseteq BA$$

Convergence under good circumstances

Safety Guards

$$\boxed{Alg'} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots} \sqsubseteq \boxed{BA}$$

Convergence under good circumstances

Luckyness via a common coin

Safety Guards

$$Alg' = \left[\boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots \right] \sqsubseteq \boxed{BA}$$

Convergence under good circumstances

Luckyness via a common coin

Eventual Synchrony + Synchronization

Safety Guards

$$\boxed{Alg'} = \boxed{\boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots \longrightarrow \boxed{GC^{R=3}} \longrightarrow \boxed{Try} \longrightarrow \dots} \sqsubseteq \boxed{BA}$$

Convergence under good circumstances

- Luckyness via a common coin
- Eventual Synchrony + Synchronization
- Unreliable Failure Detectors

Safety Guards

$$Alg' = GC^{R=3} \longrightarrow Try \longrightarrow \ldots \longrightarrow GC^{R=3} \longrightarrow Try \longrightarrow \ldots \sqsubseteq BA$$

Convergence under good circumstances

- Luckyness via a common coin
- Eventual Synchrony + Synchronization
- Unreliable Failure Detectors
- Fair scheduling / Noisy Environement

Safety Guards

$$Alg' = \left(GC^{R=3} \longrightarrow Try \longrightarrow \dots \longrightarrow GC^{R=3} \longrightarrow Try \longrightarrow \dots \right) \sqsubseteq BA$$

Convergence under good circumstances

- Luckyness via a common coin
- Eventual Synchrony + Synchronization
- Unreliable Failure Detectors
- Fair scheduling / Noisy Environement
- Synchrony + round-robin rotating leader

$$Alg' = \boxed{GC^{R=3} \longrightarrow L_i \longrightarrow \cdots \longrightarrow GC^{R=3} \longrightarrow L_j \longrightarrow \cdots} \sqsubseteq BA$$

Correct Leader

Synchrony

Positive Result 2: *There exists* a **deterministic synchronous** protocol that solves consensus, while tolerating arbitrary (**Byzantine**) failures