A belated proof of self-stabilization

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I thought that in [0] I had published three solutions, but later I learned that I had also published three problems, as the programs had been given without a demonstration of their correctness. As a partial remedy I here offer such a demonstration for the solution with three-state machines.

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I recall the construction. We consider a ring of at least 3 three-state machines. For each machine a "privilege" is defined, i.e. a boolean function of the state of that machine and of the states of that machine's immediate neighbours; when such a function is true, we say that that privilege "exists". A "move" consists of a machine observing that its privilege exists, followed by a change of that machine's state; moves of neighbours exclude each other in time. We have to show that, independently of the initial state of the whole system, within a finite number of moves (i) precisely one privilege exists and (ii) in each unbounded sequence of moves, each machine moves an unbounded number of times.

Each machine state is characterized by a variable ranging over $\{0,1,2\}$; in the descriptions of the moves, additive operations are understood to be reduced <u>mod</u> 3; the variables characterizing the left- and right-hand neighbours are denoted by L and R respectively; moves are recorded as guarded commands with the guards defining the privileges.

The ring consists "in order from left to right" of

(i) "Bottom", i.e. a machine whose state is characterized by B and whose move is given by

$$B+1=R \;\rightarrow\; B:=B+2 \qquad ;$$

(ii) one or more normal machines, characterized by a variable locally named S; their moves are

$$L=S+1 \vee S+1=R \;\rightarrow\; S:=S+1 \qquad ;$$

(iii) "Top", i.e. a machine whose state is characterized by T, whose right-hand neighbour is Bottom, and whose move is

$$L=B \wedge T\neq B+1 \;\rightarrow\; T:=B+1 \qquad .$$

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In order to demonstrate self-stabilization, we consider the string starting with B, then the S's, and ending with T. In that string we place between neighbours whose states differ an arrow such that in the direction of the arrow the state decreases (mod 3) by 1. We now give in terms of the arrows the transformations each move may effectuate; for each transformation we furthermore record the change in y, where

y = the number of left-pointing arrows + twice the number of right-pointing arrows.

For Bottom:

from	$B \leftarrow R$	to	$B \rightarrow R$	$\Delta y = +1$	(0)

For a normal machine:

from	$L \rightarrow S \quad R$	to	$L \quad S \rightarrow R$	$\Delta y = 0$	(1)
from	$L \quad S \leftarrow R$	to	$L \leftarrow S \quad R$	$\Delta y = 0$	(2)
from	$L \rightarrow S \leftarrow R$	to	$L \quad S \quad R$	$\Delta y = -3$	(3)
from	$L \rightarrow S \rightarrow R$	to	$L \quad S \leftarrow R$	$\Delta y = -3$	(4)
from	$L \leftarrow S \leftarrow R$	to	$L \rightarrow S \quad R$	$\Delta y = 0$	(5)

For Top (whose privilege also depends on B!)

from	$L \rightarrow T$	to	$L \leftarrow T$	$\Delta y = -1$	(6)
from	$L \quad T$	to	$L \leftarrow T$	$\Delta y = +1$	(7)

If the string contains 1 arrow, that one arrow remains the only one, travelling up and down the string via transformations (1) and (2), and being reflected at the ends via transformations (0) and (6). Note that, if $L \rightarrow T$ represents the only arrow, $L = B \wedge T = B-1$, and hence Top's privilege exists. We conclude from the above that it suffices to show that within a finite number of moves we have precisely 1 arrow in the string.

If the string is free of arrows, $L=B \wedge T=B$ and, in the next move, transition (7) creates the situation of precisely 1 arrow in the string.

A further consequence of the two preceding paragraphs is that under all circumstances at least one move is possible. So now we consider an unbounded sequence of moves; in the remainder we shall show that, if the number of arrows is more than 1, y will be decreased within a finite number of moves.

From the design of the move of Top we conclude

<u>Lemma 0</u> Between two successive moves of Top at least one move of Bottom takes place.

Next we establish

<u>Lemma 1</u> A sequence of moves in which Bottom does not move is finite.

<u>Proof</u> From Lemma 0 we conclude that in a sequence of moves in which Bottom does not move, at most one move of Top occurs. So it suffices to show that any sequence of normal moves is

finite. As transformations (3), (4), and (5) decrease the number of arrows and none of the normal moves increases that number, it suffices to show that a sequence of transformations (1) and (2) is finite. But this follows from the topology and the finite length of the string. (End of Proof.)

After these preparations we can now prove our

<u>Theorem</u> Within a finite number of moves, there is one arrow in the string.

<u>Proof</u> From Lemma 1 we conclude that in an unbounded sequence of moves the number of moves of Bottom is unbounded. Between two successive moves of Bottom , the situation "the left-most arrow (exists and) points to the right" is falsified at least once, a falsification that can only occur in transformation (3), (4), or (6) .

If it occurs in transformation (6), we have one arrow in the string and are done.

Otherwise it occurs in transformations (3) or (4), which decrease y by 3 ; only transformations (0) and (7) increase y but — on account of Lemma 0 — by at most 2 per move of Bottom; the nett result is that, per move of Bottom, y is decreased by at least 1 . (End of Proof.)

[0] Self-stabilizing Systems in Spite of Distributed Control, by Dijkstra, Edsger W., Comm. ACM Vol 17, nr. 11 (Nov. 1974), 643-644 .

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